

Asymptotic Approximation for the Solution to a Semi-linear Parabolic Problem in a Thick Fractal Junction

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Abstract

We consider a semi-linear parabolic problem in a model plane thick fractal junction Ω_ε , which is the union of a domain Ω_0 and a lot of joined thin trees situated ε -periodically along some interval on the boundary of Ω_0 . The trees have finite number of branching levels. The following nonlinear Robin boundary condition $\partial_\nu v_\varepsilon + \varepsilon^{\alpha_i} \kappa_i(v_\varepsilon) = \varepsilon^{\beta_i} g_\varepsilon^{(i)}$ is given on the boundaries of the branches from the i -th branching layer; α_i and β_i are real parameters. The asymptotic analysis of this problem is made as $\varepsilon \rightarrow 0$, i.e., when the number of the thin trees infinitely increases and their thickness vanishes. In particular, the corresponding homogenized problem is found and the existence and uniqueness of its solution in an anisotropic Sobolev space of multi-sheeted functions is proved. We construct the asymptotic approximation for the solution v_ε and prove the corresponding asymptotic estimate in the space $C([0, T]; L^2(\Omega_\varepsilon)) \cap L^2(0, T; H^1(\Omega_\varepsilon))$, which shows the influence of the parameters $\{\alpha_i\}$ and $\{\beta_i\}$ on the asymptotic behavior of the solution.

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Contents

1	Introduction	3
2	Statement of the problem	4
2.1	Features of the investigation	7
3	Formal asymptotic expansions for the solution	10
3.1	Outer expansions	10
3.2	Construction of inner expansions	12
3.2.1	Inner expansion in a neighborhood of I_0	12
3.2.2	Inner expansion in a neighborhood of the first branching zone I_1	13
3.2.3	Inner expansion in a neighborhood of the second branching zone I_2	15
4	Matching of asymptotic expansions and homogenized problem	16
5	Operator formulation of the homogenized problem	18
6	Asymptotic approximation	21

1 Introduction

In recent years, materials with complex structure are widely used in engineering devices in many fields of science. It is known that some properties of materials are controlled by their geometrical structure. Therefore, the study of the influence of the material microstructure can improve its useful properties and reduce undesirable effects. The main methods for this study are asymptotic methods for boundary value problems (BVP's) in domains with complex structure: perforated domains, grid-domains, domains with rapidly oscillating boundaries, thick junctions, etc.

In this paper, we begin to study asymptotic properties of solutions to BVP's in thick junctions of a new type, namely *thick fractal junctions*. A thick fractal junction is the union of some domain, which is called the junction's body, and a lot of joined thin trees situated ε -periodically along some manifold on the boundary of the junction's body. The trees have finite number of branching levels. The small parameter ε characterizes the distance between neighboring thin branches and also their thickness. On Fig. 1 you can see a heat radiator with a fractal-structure that has one branching level.



Figure 1: Heat radiator shaped like a thick fractal junction

Various constructions of thick junction type are successfully used in nanotechnologies [13], microtechnique [14], modern engineering constructions (microstrip radiator, ferrite-filled rod radiator), as well as many physical and biological systems. For example, a number of new applications are envisioned, especially regarding efficient sensors (inertial, biological, chemical), signal processing filters (ultra large band), micro-fractal constructions: fractal antennas, fractal transistors, fractal heat radiators and so on.

Such successful applications of thick-junction constructions have stimulated active learning BVP's in thick junctions with more complex structures: thick junctions with the thin junction's body [2, 3, 4], thick multi-level junctions [8, 9, 18], thick cascade junctions [5, 17], where new qualitative results were obtained. Specifically, it was shown that processes in thick multi-level junctions

behave as a many-phase system and thick cascade junctions have new kind of eigenvibrations. This means that materials with such micro-structures have some new properties.

Designing such arrays of mechanical components in thick junctions cannot be achieved with today softwares, because this would require too much CPU resources. Regarding their number of components (in some cases few thousands), development of new mathematical tools are necessary. One of them is asymptotic analysis of BVP's in thick junctions as $\varepsilon \rightarrow 0$, i.e., when the number of attached thin domains infinitely increases and their thickness decreases to zero. Asymptotic results give us the possibility to replace the original problem in a thick junction by the corresponding homogenized problem that is more simpler and then apply computer simulation. In addition, in some cases it is possible to construct accurate and numerically implementable asymptotic approximations.

As a first step, here we consider a nonlinear boundary-value problem for a reaction-diffusion equation in a model 2D thick fractal junction Ω_ε (see Fig 2). Of course, it is possible to consider a thick fractal junction that has more complex branching structures. However, the main features in the asymptotic behavior of solutions to BVP's in thick fractal junctions can be observed on the example of Ω_ε (a thick fractal junction with two branching levels).

The rest of this paper is organized as follows.

The statement of the problem and features of the investigation are given in Section 2.

In Section 3 we formally construct the leading terms of asymptotic expansions for a solution to our problem. The asymptotics consists of the outer expansions both in the junction's body and in each thin branches as well as the leading terms of inner expansions in a neighborhood both of the joint zone and each branching levels.

Then in Section 4, using the method of matched asymptotic expansions, we derive the corresponding nonstandard homogenized problem. The existence and uniqueness of its solution in an anisotropic Sobolev space of multi-sheeted functions is proved in Section 5.

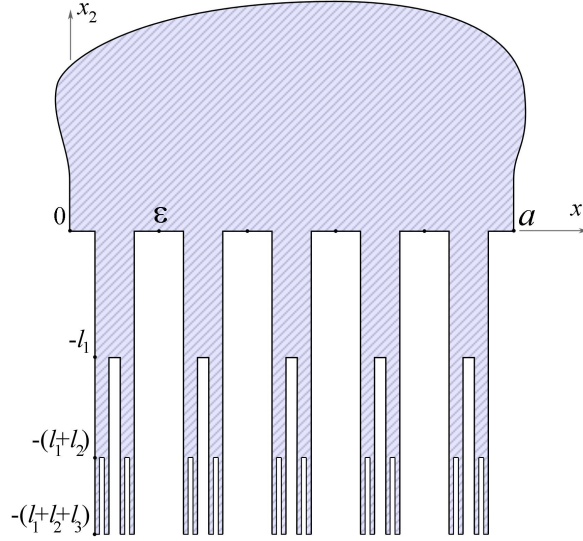
In Section 6 we construct an approximating function, find its residuals, estimate them and prove the main asymptotic estimate for the difference between the solution and the approximating function.

2 Statement of the problem

Let Ω_0 be a bounded domain in \mathbb{R}^2 with the Lipschitz boundary $\partial\Omega_0$ and $\Omega_0 \subset \{x := (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$. Let $\partial\Omega_0$ contain the segment $I_0 = \{x : x_1 \in [0, a], x_2 = 0\}$. We also assume that there exists a positive number δ_0 such that $\Omega_0 \cap \{x : 0 < x_2 < \delta_0\} = \{x : x_1 \in (0, a), x_2 \in (0, \delta_0)\}$.

Let a, l_1, l_2, l_3 be positive numbers, $h_0, h_{1,1}, h_{1,2}, h_{2,1}, h_{2,2}, h_{2,3}, h_{2,4}$ be fixed numbers from the interval $(0, 1)$ and $h_{1,1} + h_{1,2} < h_0$, $h_{2,1} + h_{2,2} < h_{1,1}$, $h_{2,3} + h_{2,4} < h_{1,2}$. Let us also introduce a small parameter $\varepsilon = \frac{a}{N}$, where N is a large positive integer.

A model thick fractal junction Ω_ε (see Fig. 2) consists of the junction's body Ω_0 ,

Figure 2: A model thick fractal junction Ω_ε

- a large number of the thin rods $G_\varepsilon^{(0)} = \bigcup_{j=1}^{N-1} G_j^{(0)}(\varepsilon)$,

$$G_j^{(0)}(\varepsilon) = \left\{ x : \left| x_1 - \varepsilon \left(j + \frac{1}{2} \right) \right| < \frac{\varepsilon h_0}{2}, \quad x_2 \in (-l_1, 0] \right\},$$

from the zero layer,

- a large number of the thin rods $G_\varepsilon^{(1,m)} = \bigcup_{j=1}^{N-1} G_j^{(1,m)}(\varepsilon)$,

$$G_j^{(1,m)}(\varepsilon) = \left\{ x : |x_1 - \varepsilon(j + b_{1,m})| < \frac{\varepsilon h_{1,m}}{2}, \quad x_2 \in (-l_2 - l_1, -l_1] \right\},$$

from the first branching layer, where $m \in \{1, 2\}$ and

$$b_{1,1} = \frac{1 - h_0 + h_{1,1}}{2}, \quad b_{1,2} = \frac{1 + h_0 - h_{1,2}}{2}, \quad (2.1)$$

- and a large number of the thin rods $G_\varepsilon^{(2,m)} = \bigcup_{j=1}^{N-1} G_j^{(2,m)}(\varepsilon)$,

$$G_j^{(2,m)}(\varepsilon) = \left\{ x : |x_1 - \varepsilon(j + b_{2,m})| < \frac{\varepsilon h_{2,m}}{2}, \quad x_2 \in (-l_3 - l_2 - l_1, -l_2 - l_1] \right\},$$

from the second branching layer, where $m \in \{1, 2, 3, 4\}$ and

$$b_{2,1} = \frac{1 - h_0 + h_{2,1}}{2}, \quad b_{2,2} = \frac{1 - h_0 + 2h_{1,1} - h_{2,2}}{2}, \quad (2.2)$$

$$b_{2,3} = \frac{1 + h_0 - 2h_{1,2} + h_{2,3}}{2}, \quad b_{2,4} = \frac{1 + h_0 - h_{2,4}}{2}. \quad (2.3)$$

Thus, $\Omega_\varepsilon = \Omega_0 \cup G_\varepsilon^{(0)} \cup G_\varepsilon^{(1)} \cup G_\varepsilon^{(2)}$, where $G_\varepsilon^{(1)} = \bigcup_{m=1}^2 G_\varepsilon^{(1,m)}$, $G_\varepsilon^{(2)} = \bigcup_{m=1}^4 G_\varepsilon^{(2,m)}$. The small parameter ε characterizes the distance between neighboring thin branches and also their

thickness. Precisely, each branch $G_j^{(i,m)}(\varepsilon)$ has small cross-section of size $\mathcal{O}(\varepsilon)$ and constant height. In addition, at fixed $j \in \{0, 1, \dots, N-1\}$ branches $G_j^{(0)}(\varepsilon)$, $\{G_j^{(1,m)}(\varepsilon)\}_{m=1}^2$, $\{G_j^{(2,m)}(\varepsilon)\}_{m=1}^4$ form the tree with two branching levels. These trees are ε -periodically distributed along the segment I_0 .

In Ω_ε we consider the following semilinear parabolic initial boundary-value problem:

$$\left\{ \begin{array}{ll} \partial_t v_\varepsilon - \Delta v_\varepsilon + k(v_\varepsilon) = f_0 & \text{in } \Omega_0 \times (0, T), \\ \partial_t v_\varepsilon - \Delta v_\varepsilon + k_i(v_\varepsilon) = 0 & \text{in } G_\varepsilon^{(i)} \times (0, T), \quad i = 0, 1, 2, \\ \partial_\nu v_\varepsilon + \varepsilon^{\alpha_i} \kappa_i(v_\varepsilon) = \varepsilon^{\beta_i} g_\varepsilon^{(i)} & \text{on } \Upsilon_\varepsilon^{(i)} \times (0, T), \quad i = 0, 1, 2, \\ \partial_\nu v_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon \setminus \left(\bigcup_{i=0}^2 \Upsilon_\varepsilon^{(i)} \right) \times (0, T), \\ [v_\varepsilon]_{x_2 = -\sum_{n=0}^i l_n} = [\partial_{x_2} v_\varepsilon]_{x_2 = -\sum_{n=0}^i l_n} = 0 & \text{on } Q_\varepsilon^{(i)} \times (0, T), \quad i = 0, 1, 2, \\ v_\varepsilon|_{t=0} = 0 & \text{in } \Omega_\varepsilon, \end{array} \right. \quad (2.4)$$

where ∂_ν is the outward normal derivative; for each index $i \in \{0, 1, 2\}$ parameters α_i and β_i are greater or equal 1, $\Upsilon_\varepsilon^{(i)} = \bigcup_{m=1}^{2i} \Upsilon_\varepsilon^{(i,m)}$, $\Upsilon_\varepsilon^{(i,m)}$ is the union of vertical boundaries of the thin rods $G_\varepsilon^{(i,m)}$, $Q_\varepsilon^{(i)} = G_\varepsilon^{(i)} \cap \{x_2 = -\sum_{n=0}^i l_n\}$, $l_0 = 0$, f_0 , $g_\varepsilon^{(i)}$, k , k_i , κ_i are given functions; the brackets denote the jump of the enclosed quantities.

Remark 2.1. Hereafter we use the following shortening: $\{x_2 = -\sum_{n=0}^i l_n\} := \{x \in \mathbb{R}^2 : x_2 = -\sum_{n=0}^i l_n\}$; also if the index $i = 0$, then the index m is absent and notation as $\Upsilon_\varepsilon^{(0,m)}$ means $\Upsilon_\varepsilon^{(0)}$.

Assumptions for the given functions are as follows. The function f_0 belongs to the space $L^2(\Omega_0 \times (0, T))$ and its support is compactly embedded in Ω_0 for a.e. $t \in (0, T)$. The functions $\{g_\varepsilon^{(i)}\}_{i=0}^2$ satisfy the following conditions:

- $g_\varepsilon^{(i)} \in L^2(D_i \times (0, T))$, where the domain

$$D_i = \left\{ x : 0 < x_1 < a, \quad -\sum_{n=0}^{i+1} l_n < x_2 < -\sum_{n=0}^i l_n \right\} \quad (2.5)$$

is filled up by the thin rods from the i -th layer in the limit passage as $\varepsilon \rightarrow 0$;

- there exist weak derivatives $\partial_{x_1} g_\varepsilon^{(i)} \in L^2(D_i \times (0, T))$, $i = 0, 1, 2$, and constants c_i , ε_0 such that for each value $\varepsilon \in (0, \varepsilon_0)$

$$\|g_\varepsilon^{(i)}\|_{L^2(D_i \times (0, T))} + \|\partial_{x_1} g_\varepsilon^{(i)}\|_{L^2(D_i \times (0, T))} \leq c_i; \quad (2.6)$$

- moreover, if $\beta_i = 1$, then there exists a function $g_0^{(i)} \in L^2(D_i \times (0, T))$ such that

$$g_\varepsilon^{(i)} \rightarrow g_0^{(i)} \quad \text{in } L^2(D_i \times (0, T)) \quad \text{as } \varepsilon \rightarrow 0. \quad (2.7)$$

The functions $k : \mathbb{R} \rightarrow \mathbb{R}$, $k_i : \mathbb{R} \rightarrow \mathbb{R}$, and $\kappa_i : \mathbb{R} \rightarrow \mathbb{R}$, $i = 0, 1, 2$ are continuously differentiable and

$$\begin{aligned} \exists c_1, c_2 > 0 : \quad c_1 \leq k' \leq c_2, \quad c_1 \leq k'_i \leq c_2, \quad c_1 \leq \kappa'_i \leq c_2 \quad \text{in } \mathbb{R}, \\ i = 0, 1, 2. \end{aligned} \quad (2.8)$$

From (2.8) it follows (see e.g. [16]) the following inequalities:

$$c_1 s^2 + k(0)s \leq k(s)s \leq c_2 s^2 + k(0)s, \quad (2.9)$$

$$\exists c_3 > 0 \quad \forall p, s \in \mathbb{R} : \quad |k(p) - k(s)| \leq c_3 |p - s|, \quad |k(s)| \leq c_3(1 + |s|) \quad (2.10)$$

(the same inequalities for the other functions $\{k_i\}, \{\kappa_i\}$).

Recall that a function $v_\varepsilon \in L^2(0, T; H^1(\Omega_\varepsilon))$, with $v'_\varepsilon \in L^2(0, T; (H^1(\Omega_\varepsilon))^*)$, is a weak solution to problem (2.4) if

$$\langle v'_\varepsilon, \psi \rangle_\varepsilon + \langle \mathcal{A}_\varepsilon(t)v_\varepsilon, \psi \rangle_\varepsilon = \langle F_\varepsilon(t), \psi \rangle_\varepsilon \quad (2.11)$$

for each $\psi \in H^1(\Omega_\varepsilon)$ and a.e. $t \in (0, T)$, and $v_\varepsilon|_{t=0} = 0$.

Here $\partial_t v_\varepsilon := v'_\varepsilon$, the brackets $\langle \cdot, \cdot \rangle_\varepsilon$ denotes the pairing of $H^1(\Omega_\varepsilon)^*$ with $H^1(\Omega_\varepsilon)$, the operator $\mathcal{A}_\varepsilon(t) : H^1(\Omega_\varepsilon) \mapsto H^1(\Omega_\varepsilon)^*$ is defined by the formula

$$\langle \mathcal{A}_\varepsilon(t)v, \psi \rangle_\varepsilon := \int_{\Omega_\varepsilon} \nabla_x v \cdot \nabla_x \psi \, dx + \int_{\Omega_0} k(v) \psi \, dx + \sum_{i=0}^2 \int_{G_\varepsilon^{(i)}} k_i(v) \psi \, dx + \varepsilon^{\alpha_i} \int_{\Upsilon_\varepsilon^{(i)}} \kappa_i(v) \psi \, dx_2$$

for all $v, \psi \in H^1(\Omega_\varepsilon)$, and the linear functional $F_\varepsilon(t) \in H^1(\Omega_\varepsilon)^*$ is defined as follows:

$$\langle F_\varepsilon(t), \psi \rangle_\varepsilon := \int_{\Omega_0} f_0 \psi \, dx + \sum_{i=0}^2 \varepsilon^{\beta_i} \int_{\Upsilon_\varepsilon^{(i)}} g_\varepsilon^{(i)} \psi \, dx_2, \quad \forall \psi \in H^1(\Omega_\varepsilon),$$

for a.e. $t \in [0, T]$. In addition, it is known that $v_\varepsilon \in C([0, T]; L^2(\Omega_\varepsilon))$ and thus the equality $v_\varepsilon|_{t=0} = 0$ makes sense.

Due to properties of the functions k, k_i, κ_i , $i = 0, 1, 2$, (see (2.8)-(2.10)) the operator \mathcal{A}_ε is bounded, strictly monotone, hemicontinuous, and coercive (we verify these properties in more detail for the corresponding homogenized operator in Section 5). Then, from well-known results of the theory of monotone operators (see e.g. [23]) it follows that for each fixed value $\varepsilon > 0$ there exists a unique weak solution to problem (2.4).

Our main research efforts are oriented towards the analytical understanding and asymptotic approximation of phenomena and processes in physics and biology which take place in thick fractal junctions involving, as models, nonlinear boundary-value problem (2.4). In particular, we want to find the corresponding homogenized problem as $\varepsilon \rightarrow 0$, to construct the asymptotic approximation for the solution v_ε and to study the influence of the parameters $\{\alpha_i\}$ and $\{\beta_i\}$ on the asymptotic behavior of the solution.

2.1 Features of the investigation

1. Thick junctions have special character of the connectedness: there are points in a thick junction, which are at a short distance of order $\mathcal{O}(\varepsilon)$, but the length of all curves, which connect these points in the junction, is order $\mathcal{O}(1)$. As a result, there are no extension operators that would be bounded uniformly in the corresponding Sobolev spaces [15]. At the same time

the availability of an uniformly bounded family of extension operators is typical supposition in overwhelming majority of the existing homogenization schemes for problems in perforated domains with the Neumann or Robin boundary conditions (see e.g. [6, 7]). In addition, thick junctions are non-convex domains with non-smooth boundaries. Therefore, solutions of boundary-value problems in such domains have only minimal H^1 -smoothness, while (see e.g. [7]) the H^2 -smoothness of a solution is necessary to prove the convergence theorem. All these factors create special difficulties in the asymptotic analysis of BVP's in thick junctions.

2. In a typical interpretation the solution to problem (2.4) denotes the density of some quantity (chemical concentration, temperature, electronic potential, etc) at equilibrium within the thick fractal junction Ω_ε . Usually for applied problems, the source of the quantity is located in the junctions body. Therefore, the right-hand side f_0 is defined in Ω_0 .
3. Standard assumptions for nonlinear terms of reaction-diffusion equations are as follows: they are Lipschitz continuous functions. This hypothesis in particular implies $|k(s)| \leq C(1 + |s|)$ for each $s \in \mathbb{R}$ and some constant C . This is enough to state that problem (2.4) has a unique solution. But, if we want to construct some approximation for a solution and to prove the corresponding estimate, we need some kind of a coercivity condition on the nonlinearity. Usually it reads as follows: $k(s)s \geq C_1|s|^2 - C_2$ for all $s \in \mathbb{R}$ and appropriate constants $C_1 > 0$, $C_2 \geq 0$.

Many physical processes, especially in chemistry and medicine, have monotonous nature. Therefore, it is naturally to impose special monotonous conditions on the nonlinear terms. In our case we propose simple conditions (2.8) that imply the coercivity conditions (2.9).

4. Asymptotic behaviour of solutions to the reaction-diffusion equation in different kind of thin domains with the uniform Neumann conditions was studied in [1, 22]. The convergence theorems were proved under the following assumptions for the nonlinear term: in [1] it is a C^2 -function with bounded derivatives and

$$\limsup_{|s| \rightarrow +\infty} \frac{k(s)}{s} < 0; \quad (2.12)$$

in [22] it is a C^1 -function, the dissipative condition (2.12) holds and

$$|k'(s)| \leq C(1 + |s|^{q-1}), \quad (2.13)$$

where $q \in (1, +\infty)$.

Let us note that the convergence theorem for the solution to our problem (2.4) can be proved under more weak assumptions for the functions $k, \{k_i\}, \{\kappa_i\}$, namely they are vanish at zero and satisfy inequality (2.13).

5. The nonlinear Robin boundary conditions are considered on the boundaries of the thin branches. These conditions mean that there is a flux of a quantity through the surfaces

of the branches. In fact very small activity holds always on the surface of some material (therefore the Robin boundary conditions are more natural for applied mathematical problems). Such semilinear boundary conditions arise in many applied problems, in particular, in the modeling of chemical reactive flows. For instance, the following function

$$\kappa(v) = \frac{\lambda v}{1 + \mu v} \quad \text{with } \lambda, \mu > 0,$$

which satisfies condition (2.8) if $f_0 \geq 0$ and $g_\varepsilon^{(i)} \equiv 0$, corresponds to the Michaelis-Menten hypothesis in biochemical reactions and to the Langmuir kinetics adsorption models (see [21, 7]).

6. In the interpretation mentioned above, the problem (2.4) describes the motion of a reactive fluid having different chemical features on different branching layers ($i = 0, 1, 2$) of the thick fractal junction. To study the influence of the boundary interactions on the asymptotic behavior of the solution, we introduce special intensity factors ε^{α_i} and ε^{β_i} in the Robin boundary conditions on the lateral sides of the thin rectangles from the i -th branching layers. The effective behavior of this reactive flow (as $\varepsilon \rightarrow 0$) is described by a new nonstandard homogenized parabolic problem containing extra zero-order terms which catch the effect of the chemical reactions depending on α_i and β_i . The asymptotic behavior of the solution is described in Theorem 6.1. Here we note only that the following differential equations

$$h_{i,m} \partial_t v_0^{(i,m)} - h_{i,m} \partial_{x_2 x_2}^2 v_0^{(i,m)} + h_{i,m} k_i(v_0^{(i,m)}) + 2\delta_{\alpha_i,1} \kappa_i(v_0^{(i,m)}) = 2\delta_{\beta_i,1} g_0^{(i)}, \quad m = \overline{1, 2i},$$

form the homogenized relations in $D_i \times (0, T)$, where $\delta_{\alpha_i,1}, \delta_{\beta_i,1}$ are Kronecker's symbols.

7. It should be stressed that the important problem for each new proposed asymptotic method is its accuracy. Therefore, the proof of the error estimate for discrepancy between the constructed approximation and the exact solution is general principle that has been applied to the analysis of the efficiency of the proposed asymptotic method. With the help of special branch-layer solutions and the method of matched asymptotic expansions, the approximation for the solution is constructed and the corresponding asymptotic error estimate in the space $C([0, T]; L^2(\Omega_\varepsilon)) \cap L^2(0, T; H^1(\Omega_\varepsilon))$ is proved in Theorem 6.1. From this theorem it follows directly the following corollary.

Corollary 2.1. *Let assumptions from Theorem 6.1 hold. Then for any $\rho \in (0, 1)$*

$$\begin{aligned} & \max_{t \in [0, T]} \left(\|v_\varepsilon(\cdot, t) - v_0^+(\cdot, t)\|_{L^2(\Omega_0)} + \sum_{i=0}^2 \sum_{m=1}^{2i} \|v_\varepsilon(\cdot, t) - v_0^{(i,m)}(\cdot, t)\|_{L^2(G_\varepsilon^{(i,m)})} \right) \\ & \leq C_0 \left(\varepsilon^{1-\rho} + \sum_{i=0}^2 \left(\varepsilon^{\alpha_i-1+\delta_{\alpha_i,1}} + (1 - \delta_{\beta_i,1}) \varepsilon^{\beta_i-1} + \delta_{\beta_i,1} \|g_\varepsilon^{(i)} - g_0^{(i)}\|_{L^2(G_\varepsilon^{(i)})} \right) \right), \end{aligned}$$

where v_ε is the solution to problem (2.4), $(v^+, v^{(0)}, \{v^{(1,m)}\}_{m=1}^2, \{v^{(2,m)}\}_{m=1}^4)$ is the multi-sheeted solution to the homogenized problem (5.1).

3 Formal asymptotic expansions for the solution

3.1 Outer expansions

Combining the algorithm of constructing asymptotics in thin domains with the methods of homogenization theory, we seek the main terms of the asymptotics for the solution v_ε in the form

$$v_\varepsilon(x, t) \approx v_0^+(x, t) + \sum_{n=1}^{+\infty} \varepsilon^n v_n^+(x, t) \quad \text{in domain } \Omega_0 \times (0, T) \quad (3.1)$$

and

$$v_\varepsilon(x, t) \approx v_0^{(i,m)}(x, t) + \sum_{n=1}^{+\infty} \varepsilon^n v_n^{(i,m)}(x, \frac{x_1}{\varepsilon} - j, t) \quad (3.2)$$

in the thin rod $G_j^{(i,m)}(\varepsilon) \times (0, T)$ from the i -th level, $j = 0, \dots, N-1$. Let us recall that $i \in \{0, 1, 2\}$ and the index $m \in \{1, 2\}$ for $i = 1$, $m \in \{1, 2, 3, 4\}$ for $i = 2$, and if $i = 0$, then m is absent and $G_j^{(0,m)}(\varepsilon) = G_j^{(0)}(\varepsilon)$ and $v_n^{(0,m)} = v_n^{(0)}$.

The asymptotic series (3.1) and (3.2) are usually called *outer expansions*.

Substituting the series (3.1) in the first equation of problem (2.4) and in the boundary conditions on $\partial\Omega_0 \setminus I_0$, collecting coefficients of the same powers of ε and taking into account the first estimate in (2.10), we get the following relations for the coefficient v_0^+ :

$$\begin{aligned} \partial_t v_0^+ - \Delta v_0^+ + k(v_0^+) &= f_0 \quad \text{in } \Omega_0 \times (0, T), \\ \partial_\nu v_0^+ &= 0 \quad \text{on } (\partial\Omega_0 \setminus I_0) \times (0, T). \end{aligned} \quad (3.3)$$

Now let us find limit relations in each domain D_i (see (2.5)). Assuming for the moment that the functions $\{v_n^{(i,m)}\}$ in (3.2) are smooth, we write their Taylor series with respect to the variable x_1 at the point $x_1 = \varepsilon(j + b_{i,m})$ (points $\{b_{i,m}\}$ are defined in (2.1)–(2.3), $b_{0,m} = b_0 = \frac{1}{2}$) and pass to the "fast" variable $\xi_1 = \varepsilon^{-1}x_1$; the indexes i, m and j are fixed. Then (3.2) takes the form

$$v_\varepsilon(x, t) \approx v_0^{(i,m)}(\varepsilon(j + b_{i,m}), x_2, t) + \sum_{n=1}^{+\infty} \varepsilon^n V_n^{(i,m,j)}(\xi_1, x_2, t), \quad (3.4)$$

where

$$\begin{aligned} V_n^{(i,m,j)}(\xi_1, x_2, t) &= v_n^{(i,m)}(\varepsilon(j + b_{i,m}), x_2, \xi_1 - j, t) \\ &+ \sum_{p=1}^n \frac{(\xi_1 - j - b_{i,m})^p}{p!} \frac{\partial^p v_{n-p}^{(i,m)}}{\partial x_1^p}(\varepsilon(j + b_{i,m}), x_2, \xi_1 - j, t). \end{aligned} \quad (3.5)$$

Let us substitute (3.4) into (2.4) instead of v_ε . Since the Laplace operator takes the form $\Delta = \varepsilon^{-2} \frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial x_2^2}$, the collection of coefficients of the same power of ε gives us one dimensional boundary value problems with respect to ξ_1 for each $t \in (0, T)$. The first problem is the following:

$$\begin{aligned} \partial_{\xi_1 \xi_1}^2 V_1^{(i,m,j)}(\xi_1, x_2, t) &= 0, \quad \xi_1 \in I_{h_{i,m}}(b_{i,m}), \\ \partial_{\xi_1} V_1^{(i,m,j)}(b_{i,m} \pm \frac{h_{i,m}}{2}, x_2, t) &= 0, \end{aligned} \quad (3.6)$$

where $\partial_{\xi_1} = \frac{\partial}{\partial \xi_1}$, $\partial_{\xi_1 \xi_1}^2 = \frac{\partial^2}{\partial \xi_1^2}$ and $I_{h_{i,m}}(b_{i,m}) = (b_{i,m} - \frac{h_{i,m}}{2}, b_{i,m} + \frac{h_{i,m}}{2})$; the variable x_2 is regarded as a parameter in this problem.

From (3.6) it follows that function $V_1^{(i,m,j)}$ doesn't depend on ξ_1 . Therefore, $V_1^{(i,m,j)}$ is equal to some function $\varphi^{(i,m)}(\varepsilon(j + b_{i,m}), x_2, t)$. Since we look only for the first terms of the asymptotics, we can regard that $\varphi^{(i,m)} \equiv 0$. Then, due to (3.5), we have

$$v_1^{(i,m)}(\varepsilon(j + b_{i,m}), x_2, \xi_1 - j, t) = -(\xi_1 - j - b_{i,m}) \partial_{x_1} v_0^{(i,m)}(\varepsilon(j + b_{i,m}), x_2, t). \quad (3.7)$$

The problem for the function $V_2^{(i,m,j)}$ is as follows:

$$-\partial_{\xi_1 \xi_1}^2 V_2^{(i,m,j)} = \left(\partial_{x_2 x_2}^2 v_0^{(i,m)} - k_i(v_0^{(i,m)}) - \partial_t v_0^{(i,m)} \right) \Big|_{x_1 = \varepsilon(j + b_{i,m})}, \quad \xi_1 \in I_{h_{i,m}}(b_{i,m}), \quad (3.8)$$

$$\partial_{\xi_1} V_2^{(i,m,j)}(\xi_1, x_2, t) \Big|_{\xi_1 = b_{i,m} \pm \frac{h_{i,m}}{2}} = \left(\mp \delta_{\alpha_i, 1} \kappa_i(v_0^{(i,m)}(x, t)) \pm \delta_{\beta_i, 1} g_0^{(i)}(x, t) \right) \Big|_{x_1 = \varepsilon(j + b_{i,m})}, \quad (3.9)$$

where $\delta_{\alpha_i, 1}, \delta_{\beta_i, 1}$ are Kronecker's symbols (recall that $\alpha_i \geq 1$ and $\beta_i \geq 1$).

The solvability condition for problem (3.8)-(3.9) is given by the differential equation

$$h_{i,m} \partial_t v_0^{(i,m)} = h_{i,m} \partial_{x_2 x_2}^2 v_0^{(i,m)} - h_{i,m} k_i(v_0^{(i,m)}) - 2\delta_{\alpha_i, 1} \kappa_i(v_0^{(i,m)}) + 2\delta_{\beta_i, 1} g_0^{(i)} \quad (3.10)$$

with respect to variables x_2 and t at the fixed value of $x_1 = \varepsilon(j + b_{i,m})$.

Since the points $\{x_1 = \varepsilon(j + b_{i,m}) : j = 0, \dots, N-1\}$ form the ε -net in the interval $(0, a)$, we can extend all equations obtained above on N segments to the rectangle D_i ($i = 0, 1, 2$). Thus, we get the following differential equation

$$h_0 \partial_t v_0^{(0)} = h_0 \partial_{x_2 x_2}^2 v_0^{(0)} - h_0 k_0(v_0^{(0)}) - 2\delta_{\alpha_0, 1} \kappa_0(v_0^{(0)}) + 2\delta_{\beta_0, 1} g_0^{(0)} \quad (3.11)$$

in $D_0 \times (0, T)$ ($h_{0,m} = h_0$); we get two differential equations ($m=1, 2$)

$$h_{1,m} \partial_t v_0^{(1,m)} = h_{1,m} \partial_{x_2 x_2}^2 v_0^{(1,m)} - h_{1,m} k_1(v_0^{(1,m)}) - 2\delta_{\alpha_1, 1} \kappa_1(v_0^{(1,m)}) + 2\delta_{\beta_1, 1} g_0^{(1)} \quad (3.12)$$

in $D_1 \times (0, T)$; and we get four differential equations ($m=1, 2, 3, 4$)

$$h_{2,m} \partial_t v_0^{(2,m)} = h_{2,m} \partial_{x_2 x_2}^2 v_0^{(2,m)} - h_{2,m} k_2(v_0^{(2,m)}) - 2\delta_{\alpha_2, 1} \kappa_2(v_0^{(2,m)}) + 2\delta_{\beta_2, 1} g_0^{(2)} \quad (3.13)$$

in $D_2 \times (0, T)$. Here the variable x_1 is regarded as a parameter.

If we substitute (3.4) for $i = 2$ into the Neumann condition on the bases

$$Q_\varepsilon^{(3)} = \overline{\Omega}_\varepsilon \cap \{x : x_2 = -(l_1 + l_2 + l_3)\}$$

and taking again that the points $\{x_1 = \varepsilon(j + b_{2,m}) : j = 0, \dots, N-1\}$ form the ε -net in the interval $(0, a)$ in account, we obtain the following boundary conditions for functions $\{v_0^{(2,m)}\}$:

$$\partial_{x_2} v_0^{(2,m)}(x_1, -(l_1 + l_2 + l_3), t) = 0, \quad m = 1, 2, 3, 4. \quad (3.14)$$

To find transmission conditions on the joint zone I_0 and on each branching zones $I_1 = \{x : x_1 \in (0, a), x_2 = -l_1\}$, $I_2 = \{x : x_1 \in (0, a), x_2 = -(l_1 + l_2)\}$, we use the method of matched asymptotic expansions for the outer expansions (3.1), (3.2) and inner expansions in neighborhoods of I_0, I_1 and I_2 .

3.2 Construction of inner expansions

3.2.1 Inner expansion in a neighborhood of I_0

In a neighborhood of the joint zone I_0 we introduce the "rapid" coordinates $\xi = (\xi_1, \xi_2)$, where $\xi_1 = \varepsilon^{-1}x_1$ and $\xi_2 = \varepsilon^{-1}x_2$. Passing to $\varepsilon = 0$, we see that the rod $G_0^{(0)}(\varepsilon)$ transforms into the semi-infinite strip

$$\Pi_{h_0}^- = \left(\frac{1}{2} - \frac{h_0}{2}, \frac{1}{2} + \frac{h_0}{2}\right) \times (-\infty, 0];$$

the domain Ω_0 transforms into the first quadrant $\{\xi : \xi_1 > 0, \xi_2 > 0\}$. Taking into account the periodic structure of Ω_ε in a neighborhood of I_0 , we take the following cell of periodicity

$$\Pi_0 = \Pi_{h_0}^- \cup \Pi^+$$

(see Fig. 3), where junction-layer problems will be considered; here $\Pi^+ = (0, 1) \times (0, +\infty)$. Obviously, solutions of these joint-layer problems must be 1-periodic in ξ_1 , i.e.,

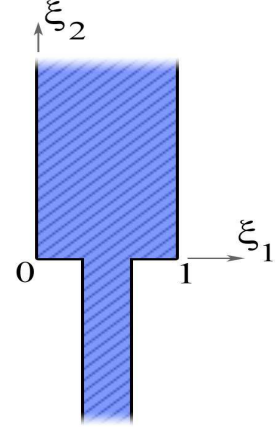


Figure 3: The cell of periodicity Π_0

$$\partial_{\xi_1}^p Z(\xi)|_{\xi_1=0} = \partial_{\xi_1}^p Z(\xi)|_{\xi_1=1}, \quad \xi \in \partial\Pi^+, \quad \xi_2 > 0, \quad p = 0, 1. \quad (3.15)$$

We propose the following ansatz for the inner asymptotic expansion in a neighborhood of $I_0 \cap \Omega_\varepsilon$:

$$v_\varepsilon \approx v_0^+(x_1, 0, t) + \varepsilon \left(Z_1^{(0)}\left(\frac{x}{\varepsilon}\right) \partial_{x_1} v_0^+(x_1, 0, t) + Z_2^{(0)}\left(\frac{x}{\varepsilon}\right) \partial_{x_2} v_0^+(x_1, 0, t) \right) + \dots \quad (3.16)$$

Substituting (3.16) in the differential equations of problem (2.4) and in the corresponding boundary conditions, taking into account that the Laplace operator takes the following form $\varepsilon^{-2}\Delta_\xi$ in the coordinates ξ and collecting the coefficients of the same power of ε , we arrive the following junction-layer problems for the coefficients $Z_1^{(0)}$ and $Z_2^{(0)}$ (to these problems we must add the periodic conditions (3.15)):

$$\begin{aligned} -\Delta_\xi Z_p^{(0)}(\xi) &= 0, & \xi \in \Pi_0, \\ \partial_{\xi_2} Z_p^{(0)}(\xi_1, 0) &= 0, & \xi_1 \in (0, 1) \setminus \left(\frac{1}{2} - \frac{h_0}{2}, \frac{1}{2} + \frac{h_0}{2}\right), \\ \partial_{\xi_1} Z_p^{(0)}(\xi) &= -\delta_{p,1}, & \xi \in \partial\Pi_{h_1}^- \cap \{\xi : \xi_2 < 0\}, \quad p = 1, 2. \end{aligned} \quad (3.17)$$

The existence and the main asymptotic relations for solutions of problems (3.17) can be obtained from general results about the asymptotic behavior of solutions to elliptic problems in domains with different exits to infinity [11, 20]. However, if a domain, where we consider a boundary-value problem, has some symmetry, then we can define more exactly the asymptotic relations and detect other properties of junction-layer solutions (see Lemma 4.1 and Corollary 4.1 from [15], see also [19]). From those results it follows the following proposition.

Proposition 3.1. *There exist unique solutions $Z_1^{(0)}, Z_2^{(0)} \in H_{loc, \xi_2}^1(\Pi_0)$ to problems (3.17) respectively, which have the following differentiable asymptotics*

$$Z_1^{(0)}(\xi) = \begin{cases} \mathcal{O}(\exp(-2\pi\xi_2)), & \xi_2 \rightarrow +\infty, \\ \left(-\xi_1 + \frac{1}{2}\right) + \mathcal{O}(\exp(\pi h_0^{-1}\xi_2)), & \xi_2 \rightarrow -\infty, \end{cases} \quad (3.18)$$

$$Z_2^{(0)}(\xi) = \begin{cases} \xi_2 + \mathcal{O}(\exp(-2\pi\xi_2)), & \xi_2 \rightarrow +\infty, \\ \frac{\xi_2}{h_0} + C_2 + \mathcal{O}(\exp(\pi h_0^{-1}\xi_2)), & \xi_2 \rightarrow -\infty, \end{cases} \quad (3.19)$$

Moreover, function $Z_1^{(0)}$ is odd in ξ_1 and function $Z_2^{(0)}$ is even in ξ_1 with respect to $\frac{1}{2}$.

Recall that a function Z belongs to the Sobolev space $H_{loc, \xi_2}^1(\Pi_0)$ if for every $R > 0$ this function $Z \in H^1(\Pi_0 \cap \{\xi : |\xi_2| < R\})$.

3.2.2 Inner expansion in a neighborhood of the first branching zone I_1

In a neighborhood of I_1 we introduce the "rapid" coordinates $\xi = (\xi_1, \xi_2)$, where $\xi_1 = \varepsilon^{-1}x_1$ and $\xi_2 = \varepsilon^{-1}(x_2 + l_1)$. Passing to $\varepsilon = 0$, we see that the rod $G_0^{(0)}(\varepsilon)$ transforms into the semi-infinite strip

$$\Pi_{h_0}^+ = \left(\frac{1}{2} - \frac{h_0}{2}, \frac{1}{2} + \frac{h_0}{2}\right) \times (0, +\infty)$$

and rods $G_0^{(1,m)}(\varepsilon)$, $m = 1, 2$, transform into the semi-infinite strips

$$\Pi_{1,m}^- = \left(b_{1,m} - \frac{h_{1,m}}{2}, b_{1,m} + \frac{h_{1,m}}{2}\right) \times (-\infty, 0], \quad m = 1, 2,$$

respectively. Taking into account the periodic structure of Ω_ε in a neighborhood of I_1 , we take the following cell of periodicity

$$\Pi_1 = \Pi_{h_0}^+ \cup \Pi_{1,1}^- \cup \Pi_{1,2}^-,$$

where branch-layer problems will be considered.

Now we propose the following ansatz for the inner asymptotic expansion in a neighborhood of $I_1 \cap (G_\varepsilon^{(0)} \cup G_\varepsilon^{(1)})$:

$$\begin{aligned} v_\varepsilon(x, t) \approx & v_0^{(0)}(x_1, -l_1, t) + \varepsilon \left(Z_1^{(1)}\left(\frac{x_1}{\varepsilon}, \frac{x_2+l_1}{\varepsilon}\right) \partial_{x_1} v_0^{(0)}(x_1, -l_1, t) \right. \\ & \left. + \left\{ \eta_1(x_1, t) \Xi_1^{(1)}\left(\frac{x_1}{\varepsilon}, \frac{x_2+l_1}{\varepsilon}\right) + (1 - \eta_1(x_1, t)) \Xi_2^{(1)}\left(\frac{x_1}{\varepsilon}, \frac{x_2+l_1}{\varepsilon}\right) \right\} \partial_{x_2} v_0^{(0)}(x_1, -l_1, t) \right) + \dots \end{aligned} \quad (3.20)$$

where $Z_1^{(1)}(\xi)$, $\Xi_1^{(1)}(\xi)$, $\Xi_2^{(1)}(\xi)$, $\xi \in \Pi_1$, are solutions to branch-layer problems, which 1-periodic extended along the coordinate axis O_{ξ_1} , the function η_1 will be defined from matching conditions.

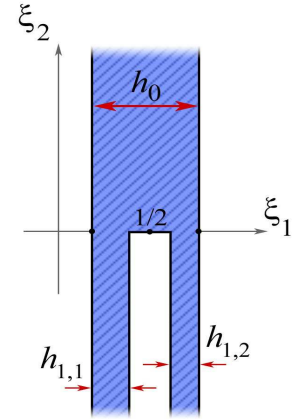


Figure 4: The cell of periodicity Π_0

Substituting (3.20) in the corresponding differential equation of problem (2.4) and boundary conditions, we arrive branch-layer problems for the functions $Z_1^{(1)}$, $\Xi_1^{(1)}$, $\Xi_2^{(1)}$. So, the functions $\Xi_1^{(1)}$ and $\Xi_2^{(1)}$ are solution to the following homogeneous problem

$$\begin{aligned} -\Delta_\xi \Xi(\xi) &= 0, & \xi \in \Pi_1, \\ \partial_{\xi_1} \Xi(\xi) &= 0, & \xi \in \partial_{\parallel} \Pi_1, \\ \partial_{\xi_2} \Xi(\xi_1, 0) &= 0, & (\xi_1, 0) \in \partial \Pi_1 \setminus \partial_{\parallel} \Pi_1, \end{aligned} \quad (3.21)$$

where $\partial_{\parallel} \Pi_1$ is the union of the vertical sides of $\partial \Pi_1$. Again using approach mentioned above, we conclude.

Proposition 3.2. *There exist two solutions $\Xi_1, \Xi_2 \in H_{loc, \xi_2}^1(\Pi_1)$ to problems (3.21), which have the following differentiable asymptotics:*

$$\Xi_1(\xi) = \begin{cases} \xi_2 + \mathcal{O}(\exp(-\frac{\pi \xi_2}{h_0})), & \xi_2 \rightarrow +\infty, \xi \in \Pi_{h_0}^+, \\ \frac{h_0}{h_{1,1}} \xi_2 + C_1^{(1)} + \mathcal{O}(\exp(\frac{\pi \xi_2}{h_{1,1}})), & \xi_2 \rightarrow -\infty, \xi \in \Pi_{1,1}^-, \\ C_2^{(1)} + \mathcal{O}(\exp(\frac{\pi \xi_2}{h_{1,2}})), & \xi_2 \rightarrow -\infty, \xi \in \Pi_{1,2}^-, \end{cases} \quad (3.22)$$

$$\Xi_2(\xi) = \begin{cases} \xi_2 + \mathcal{O}(\exp(-\frac{\pi \xi_2}{h_0})), & \xi_2 \rightarrow +\infty, \xi \in \Pi_{h_0}^+, \\ C_1^{(2)} + \mathcal{O}(\exp(\frac{\pi \xi_2}{h_{1,1}})), & \xi_2 \rightarrow -\infty, \xi \in \Pi_{1,1}^-, \\ \frac{h_0}{h_{1,2}} \xi_2 + C_2^{(2)} + \mathcal{O}(\exp(\frac{\pi \xi_2}{h_{1,2}})), & \xi_2 \rightarrow -\infty, \xi \in \Pi_{1,2}^-, \end{cases} \quad (3.23)$$

where $C_1^{(1)}, C_2^{(1)}, C_1^{(2)}, C_2^{(2)}$ are some fixed constants.

Any another solution to the homogeneous problem (3.21), which has polynomial grow at infinity, can be presented as a linear combination $c_0 + c_1 \Xi_1 + c_2 \Xi_2$.

The function $Z_1^{(1)}$ is a solution to the following problem:

$$\begin{aligned} -\Delta_\xi Z(\xi) &= 0, & \xi \in \Pi_1, \\ \partial_{\xi_1} Z(\xi) &= -1, & \xi \in \partial_{\parallel} \Pi_1, \\ \partial_{\xi_2} Z(\xi_1, 0) &= 0, & (\xi_1, 0) \in \partial \Pi_1 \setminus \partial_{\parallel} \Pi_1. \end{aligned} \quad (3.24)$$

Proposition 3.3. *There exists the unique solution $Z \in H_{loc, \xi_2}^1(\Pi_0)$ to problems (3.24), which has the following differentiable asymptotics:*

$$Z(\xi) = \begin{cases} -\xi_1 + \frac{1}{2} + \mathcal{O}(\exp(-\frac{\pi \xi_2}{h_0})), & \xi_2 \rightarrow +\infty, \xi \in \Pi_{h_0}^+, \\ -\xi_1 + b_{1,1} + C_1 + \mathcal{O}(\exp(\frac{\pi \xi_2}{h_{1,1}})), & \xi_2 \rightarrow -\infty, \xi \in \Pi_{1,1}^-, \\ -\xi_1 + b_{1,2} + C_2 + \mathcal{O}(\exp(\frac{\pi \xi_2}{h_{1,2}})), & \xi_2 \rightarrow -\infty, \xi \in \Pi_{1,2}^-, \end{cases} \quad (3.25)$$

where C_1, C_2 are some fixed constants.

Thus, we set $\Xi_1^{(1)} = \Xi_1$, $\Xi_2^{(1)} = \Xi_2$ and $Z_1^{(1)} = Z$.

3.2.3 Inner expansion in a neighborhood of the second branching zone I_2

In a neighborhood of I_2 we introduce the "rapid" coordinates $\xi = (\xi_1, \xi_2)$, where $\xi_1 = \varepsilon^{-1}x_1$ and $\xi_2 = \varepsilon^{-1}(x_2 + l_1 + l_2)$. Passing to $\varepsilon = 0$, we see that the rods $G_0^{(1,m)}(\varepsilon)$, $m = 1, 2$, transform into the semi-infinite strips $\Pi_{1,m}^+ = (b_{1,m} - \frac{h_{1,m}}{2}, b_{1,m} + \frac{h_{1,m}}{2}) \times (0, +\infty)$, $m = 1, 2$, respectively, and the rods $G_0^{(2,m)}(\varepsilon)$, $m = 1, 2, 3, 4$, transform into the semi-infinite strips $\Pi_{2,m}^- = (b_{2,m} - \frac{h_{2,m}}{2}, b_{2,m} + \frac{h_{2,m}}{2}) \times (-\infty, 0]$, $m = 1, 2, 3, 4$, respectively.

Taking into account the periodic structure of Ω_ε in a neighborhood of I_2 , we take the following two cells of periodicity

$$\Pi_2^{(1)} = \Pi_{1,1}^+ \cup \Pi_{2,1}^- \cup \Pi_{2,2}^- \quad \text{and} \quad \Pi_2^{(2)} = \Pi_{1,2}^+ \cup \Pi_{2,3}^- \cup \Pi_{2,4}^-$$

where branch-layer problems will be considered.

Now we propose the following two inner asymptotic expansions in a neighborhood of $I_2 \cap (G_\varepsilon^{(1)} \cup G_\varepsilon^{(2)})$, namely the first one is as follows:

$$\begin{aligned} v_\varepsilon(x, t) &\approx v_0^{(1,1)}(x_1, 0, t) + \varepsilon \left(Z_1^{(2,1)}\left(\frac{x_1}{\varepsilon}, \frac{x_2+l_1+l_2}{\varepsilon}\right) \partial_{x_1} v_0^{(1,1)}(x_1, 0, t) \right. \\ &\quad \left. + \left\{ \eta_{2,1}(x_1, t) \Xi_1^{(2,1)}\left(\frac{x_1}{\varepsilon}, \frac{x_2+l_1+l_2}{\varepsilon}\right) + (1 - \eta_{2,1}(x_1, t)) \Xi_2^{(2,1)}\left(\frac{x_1}{\varepsilon}, \frac{x_2+l_1+l_2}{\varepsilon}\right) \right\} \partial_{x_2} v_0^{(1,1)}(x_1, 0, t) \right) + \dots \end{aligned} \quad (3.26)$$

in a neighborhood of $I_2 \cap \left(G_\varepsilon^{(1,1)} \cup \left(\bigcup_{m=1}^2 G_\varepsilon^{(2,m)} \right) \right)$, and the second one

$$\begin{aligned} v_\varepsilon(x, t) &\approx v_0^{(1,2)}(x_1, 0, t) + \varepsilon \left(Z_1^{(2,2)}\left(\frac{x_1}{\varepsilon}, \frac{x_2+l_1+l_2}{\varepsilon}\right) \partial_{x_1} v_0^{(1,2)}(x_1, 0, t) \right. \\ &\quad \left. + \left\{ \eta_{2,2}(x_1, t) \Xi_1^{(2,2)}\left(\frac{x_1}{\varepsilon}, \frac{x_2+l_1+l_2}{\varepsilon}\right) + (1 - \eta_{2,2}(x_1, t)) \Xi_2^{(2,2)}\left(\frac{x_1}{\varepsilon}, \frac{x_2+l_1+l_2}{\varepsilon}\right) \right\} \partial_{x_2} v_0^{(1,2)}(x_1, 0, t) \right) + \dots \end{aligned} \quad (3.27)$$

in a neighborhood of $I_2 \cap \left(G_\varepsilon^{(1,2)} \cup \left(\bigcup_{m=3}^4 G_\varepsilon^{(2,m)} \right) \right)$.

Coefficients $Z_1^{(2,1)}(\xi)$, $\Xi_1^{(2,1)}(\xi)$, $\Xi_2^{(2,1)}(\xi)$ ($\xi \in \Pi_2^{(1)}$) in (3.26) and coefficients $Z_1^{(2,2)}(\xi)$, $\Xi_1^{(2,2)}(\xi)$, $\Xi_2^{(2,2)}(\xi)$ ($\xi \in \Pi_2^{(2)}$) in (3.27) are solutions to branch-layer problems, which 1-periodic extended along the coordinate axis O_{ξ_1} ; the functions $\eta_{2,1}$ and $\eta_{2,2}$ will be defined from matching conditions.

Namely, $Z_1^{(2,1)}$ and $Z_1^{(2,2)}$ are solutions to problem (3.24) but now in $\Pi_2^{(1)}$ and $\Pi_2^{(2)}$ respectively. Applying results of Proposition 3.3, we can state that there exist the unique solutions with the following differentiable asymptotics:

$$Z_1^{(2,1)}(\xi) = \begin{cases} -\xi_1 + b_{1,1} + \mathcal{O}\left(\exp\left(-\frac{\pi\xi_2}{h_{1,1}}\right)\right), & \xi_2 \rightarrow +\infty, \xi \in \Pi_{1,1}^+, \\ -\xi_1 + b_{2,1} + C_1^{(3)} + \mathcal{O}\left(\exp\left(\frac{\pi\xi_2}{h_{2,1}}\right)\right), & \xi_2 \rightarrow -\infty, \xi \in \Pi_{2,1}^-, \\ -\xi_1 + b_{2,2} + C_2^{(3)} + \mathcal{O}\left(\exp\left(\frac{\pi\xi_2}{h_{2,2}}\right)\right), & \xi_2 \rightarrow -\infty, \xi \in \Pi_{2,2}^-, \end{cases} \quad (3.28)$$

$$Z_1^{(2,2)}(\xi) = \begin{cases} -\xi_1 + b_{1,2} + \mathcal{O}\left(\exp\left(-\frac{\pi\xi_2}{h_{1,2}}\right)\right), & \xi_2 \rightarrow +\infty, \xi \in \Pi_{1,2}^+, \\ -\xi_1 + b_{2,3} + C_1^{(4)} + \mathcal{O}\left(\exp\left(\frac{\pi\xi_2}{h_{2,3}}\right)\right), & \xi_2 \rightarrow -\infty, \xi \in \Pi_{2,3}^-, \\ -\xi_1 + b_{2,4} + C_2^{(4)} + \mathcal{O}\left(\exp\left(\frac{\pi\xi_2}{h_{2,4}}\right)\right), & \xi_2 \rightarrow -\infty, \xi \in \Pi_{2,4}^-. \end{cases} \quad (3.29)$$

Functions $\Xi_1^{(2,1)}, \Xi_2^{(2,1)}$ and $\Xi_1^{(2,2)}, \Xi_2^{(2,2)}$ are solutions to problem (3.21) but now in $\Pi_2^{(1)}$ and $\Pi_2^{(2)}$ respectively. From Proposition 3.2 it follows that they have the corresponding differentiable asymptotics (3.22) and (3.23).

4 Matching of asymptotic expansions and homogenized problem

We have formally constructed the leading terms of the asymptotic expansions constructed in subsections 3.1 and 3.2 in different parts of the thick fractal junction Ω_ε . Next we apply the method of matched asymptotic expansions [10] to complete the constructions. Following this method, the asymptotics of the leading terms of outer expansions (3.1) and (3.2) as $x_2 \rightarrow \pm \sum_{p=0}^m l_p$, $m = 0, 1, 2$, have to coincide with the corresponding asymptotics of the inner expansions (3.16), (3.20), (3.26) and (3.27) as $\eta_2 \rightarrow \pm\infty$ respectively.

Near the point $(\varepsilon(j + \frac{1}{2}), 0) \in I_0$ at the fixed value of t , the function v_0^+ has the following asymptotics

$$v_0^+(\varepsilon(j + \frac{1}{2}), 0, t) + \varepsilon \xi_2 \partial_{x_2} v_0^+(\varepsilon(j + \frac{1}{2}), 0, t) + \dots \quad \text{as } x_2 \rightarrow 0 + 0.$$

Taking into account the asymptotics of $Z_1^{(0)}$ and $Z_2^{(0)}$ as $\xi_2 \rightarrow +\infty$ (see (3.18) and (3.19)), we conclude that the matching conditions are satisfied for the expansion (3.1) and (3.16).

The asymptotics of the outer expansion (3.2) is equal to

$$v_0^{(0)}(\varepsilon(j + \frac{1}{2}), 0, t) + \varepsilon \left((-\xi_1 + \frac{1}{2} + j) \partial_{x_1} v_0^{(0)}(\varepsilon(j + \frac{1}{2}), 0, t) + \xi_2 \partial_{x_2} v_0^{(0)}(\varepsilon(j + \frac{1}{2}), 0, t) \right) + \dots \quad (4.1)$$

as $x_2 \rightarrow 0 - 0$, $(x, t) \in G_j^{(0)}(\varepsilon) \times (0, T)$. Keeping in mind the asymptotics of functions $Z_1^{(0)}$ and $Z_2^{(0)}$ as $\xi_2 \rightarrow -\infty$, we find the asymptotics of the leading terms of inner expansion (3.16)

$$v_0^+(\varepsilon(j + \frac{1}{2}), 0, t) + \varepsilon \left((-\xi_1 + j + \frac{1}{2}) \partial_{x_1} v_0^+(\varepsilon(j + \frac{1}{2}), 0, t) + (\frac{\xi_2}{h_0} + C_2) \partial_{x_2} v_0^+(\varepsilon(j + \frac{1}{2}), 0, t) \right) + \dots \quad (4.2)$$

as $\xi_2 \rightarrow -\infty$, $\xi \in \Pi_{h_0}^-$. Comparing terms of (4.1) and (4.2) at ε^0 and ε respectively, we conclude that matching conditions are satisfied if

$$v_0^+(\varepsilon(j + \frac{1}{2}), 0, t) = v_0^{(0)}(\varepsilon(j + \frac{1}{2}), 0, t), \quad \partial_{x_2} v_0^+(\varepsilon(j + \frac{1}{2}), 0, t) = h_0 \partial_{x_2} v_0^{(0)}(\varepsilon(j + \frac{1}{2}), 0, t),$$

$j = 0, 1, \dots, N-1$. Since the points $\{x_1 = \varepsilon(j + \frac{1}{2}) : j = 0, \dots, N-1\}$ form the ε -net in the interval $(0, a)$, we can spread these relations into all interval I_0 and get the first transmission conditions

$$v_0^+(x_1, 0, t) = v_0^{(0)}(x_1, 0, t), \quad (x_1, t) \in (0, a) \times (0, T), \quad (4.3)$$

$$\partial_{x_2} v_0^+(x_1, 0, t) = h_0 \partial_{x_2} v_0^{(0)}(x_1, 0, t), \quad (x_1, t) \in (0, a) \times (0, T). \quad (4.4)$$

Now we verify matching conditions at the point $(\varepsilon(j + \frac{1}{2}), -l_1) \in I_1$. It is easy to see that they are satisfied for the expansion (3.2) as $x_2 \rightarrow -l_1 + 0$ ($x \in G_j^{(0)}(\varepsilon)$) and for the expansion (3.20) as $\xi_2 \rightarrow +\infty$ ($\xi \in \Pi_{h_0}^+$).

Bearing in mind (3.22), (3.23) and (3.25), we find at fixed value of $t \in (0, T)$ the following asymptotics of (3.20):

$$\begin{aligned} & v_0^{(0)}(\varepsilon(j + b_{1,1}), -l_1, t) + \varepsilon \left((-\xi_1 + j + b_{1,1} + C_1) \partial_{x_1} v_0^{(0)}(\varepsilon(j + b_{1,1}), -l_1, t) \right. \\ & \left. + \left\{ \eta_1(\varepsilon(j + b_{1,1}), t) \left(\frac{h_0}{h_{1,1}} \xi_2 + C_1^{(1)} \right) + (1 - \eta_1(\varepsilon(j + b_{1,1}), t)) C_1^{(2)} \right\} \partial_{x_2} v_0^{(0)}(\varepsilon(j + b_{1,1}), -l_1, t) \right) + \dots \\ & \text{as } \xi_2 \rightarrow -\infty, \quad \xi \in \Pi_{1,1}^-, \end{aligned} \quad (4.5)$$

and

$$\begin{aligned} & v_0^{(0)}(\varepsilon(j + b_{1,2}), -l_1, t) + \varepsilon \left((-\xi_1 + j + b_{1,2} + C_2) \partial_{x_1} v_0^{(0)}(\varepsilon(j + b_{1,2}), -l_1, t) \right. \\ & \left. + \left\{ (1 - \eta_1(\varepsilon(j + b_{1,2}), t)) \left(\frac{h_0}{h_{1,2}} \xi_2 + C_2^{(2)} \right) + \eta_1(\varepsilon(j + b_{1,2}), t) C_2^{(1)} \right\} \partial_{x_2} v_0^{(0)}(\varepsilon(j + b_{1,2}), -l_1, t) \right) + \dots \\ & \text{as } \xi_2 \rightarrow -\infty, \quad \xi \in \Pi_{1,2}^-. \end{aligned} \quad (4.6)$$

Asymptotic forms of outer expansions (3.2) at $i = 1$ and $m = 1, 2$ are equal to

$$\begin{aligned} & v_0^{(1,1)}(\varepsilon(j + b_{1,1}), -l_1, t) + \varepsilon \left((-\xi_1 + b_{1,1} + j) \partial_{x_1} v_0^{(1,1)}(\varepsilon(j + b_{1,1}), -l_1, t) \right. \\ & \left. + \xi_2 \partial_{x_2} v_0^{(1,1)}(\varepsilon(j + b_{1,1}), -l_1, t) \right) + \dots \end{aligned} \quad (4.7)$$

as $x_2 \rightarrow -l_1 - 0$, $x \in G_j^{(1,1)}(\varepsilon)$, and

$$\begin{aligned} & v_0^{(1,2)}(\varepsilon(j + b_{1,2}), -l_1, t) + \varepsilon \left((-\xi_1 + b_{1,2} + j) \partial_{x_1} v_0^{(1,2)}(\varepsilon(j + b_{1,2}), -l_1, t) \right. \\ & \left. + \xi_2 \partial_{x_2} v_0^{(1,2)}(\varepsilon(j + b_{1,2}), -l_1, t) \right) + \dots \end{aligned} \quad (4.8)$$

as $x_2 \rightarrow -l_1 - 0$, $x \in G_j^{(1,2)}(\varepsilon)$.

To satisfy the matching conditions, we compare terms of (4.5) and (4.7), (4.6) and (4.8) at ε^0 and ε^1 . As a result, we get

$$\begin{aligned} & v_0^{(0)}(\varepsilon(j + b_{1,m}), -l_1, t) = v_0^{(1,m)}(\varepsilon(j + b_{1,m}), -l_1, t), \quad m = 1, 2, \\ & \eta_1(\varepsilon(j + b_{1,1}), t) h_0 \partial_{x_2} v_0^{(0)}(\varepsilon(j + b_{1,1}), -l_1, t) = h_{1,1} \partial_{x_2} v_0^{(1,1)}(\varepsilon(j + b_{1,1}), -l_1, t), \\ & (1 - \eta_1(\varepsilon(j + b_{1,2}), t)) h_0 \partial_{x_2} v_0^{(0)}(\varepsilon(j + b_{1,2}), -l_1, t) = h_{1,2} \partial_{x_2} v_0^{(1,2)}(\varepsilon(j + b_{1,2}), -l_1, t), \end{aligned}$$

for $j = 0, 1, \dots, N - 1$. Since the sets $\{x_1 = \varepsilon(j + b_{1,1}) : j = 0, \dots, N - 1\}$ $\{x_1 = \varepsilon(j + b_{1,2}) : j = 0, \dots, N - 1\}$ form the ε -net in the interval $(0, a)$, we can spread these relations into all interval I_1 and deduce the second transmission conditions

$$v_0^{(0)}(x_1, -l_1, t) = v_0^{(1,m)}(x_1, -l_1, t), \quad m = 1, 2, \quad (4.9)$$

$$h_0 \partial_{x_2} v_0^{(0)}(x_1, -l_1, t) = h_{1,1} \partial_{x_2} v_0^{(1,1)}(x_1, -l_1, t) + h_{1,2} \partial_{x_2} v_0^{(1,2)}(x_1, -l_1, t) \quad (4.10)$$

and determine the function

$$\eta_1(x_1, t) := \frac{h_{1,1} \partial_{x_2} v_0^{(1,1)}(x_1, -l_1, t)}{h_{1,1} \partial_{x_2} v_0^{(1,1)}(x_1, -l_1, t) + h_{1,2} \partial_{x_2} v_0^{(1,2)}(x_1, -l_1, t)} \quad (4.11)$$

for $x_1 \in (0, a)$ and $t \in (0, T)$.

Due to (4.9)

$$(-\xi_1 + j + b_{1,m}) \partial_{x_1} v_0^{(0)}(\varepsilon(j + b_{1,m}), -l_1, t) = (-\xi_1 + j + b_{1,m}) \partial_{x_1} v_0^{(1,m)}(\varepsilon(j + b_{1,m}), -l_1, t), \quad m = 1, 2.$$

Therefore, the matching conditions are satisfied for the leading terms of asymptotic expansions (3.2) and (3.20) at each point $(\varepsilon(j + \frac{1}{2}), -l_1) \in I_1$, $j = 0, 1, \dots, N - 1$, if (4.9), (4.10) and (4.11) hold.

In analogous way we can deduce the following two kinds of transmission conditions at $x_2 = -(l_1 + l_2)$:

$$v_0^{(1,1)} = v_0^{(2,1)} = v_0^{(2,2)} \quad \text{on } I_2 \times (0, T), \quad (4.12)$$

$$h_{1,1} \partial_{x_2} v_0^{(1,1)} = h_{2,1} \partial_{x_2} v_0^{(2,1)} + h_{2,2} \partial_{x_2} v_0^{(2,2)} \quad \text{on } I_2 \times (0, T), \quad (4.13)$$

and

$$v_0^{(1,2)} = v_0^{(2,3)} = v_0^{(2,4)} \quad \text{on } I_2 \times (0, T), \quad (4.14)$$

$$h_{1,2} \partial_{x_2} v_0^{(1,2)} = h_{2,3} \partial_{x_2} v_0^{(2,3)} + h_{2,4} \partial_{x_2} v_0^{(2,4)} \quad \text{on } I_2 \times (0, T). \quad (4.15)$$

In addition, the functions $\eta_{2,1}$ and $\eta_{2,2}$ in (3.26) and (3.27) are defined by formulas

$$\eta_{2,1}(x_1, t) = \frac{h_{2,1} \partial_{x_2} v_0^{(2,1)}(x_1, -(l_1 + l_2), t)}{h_{2,1} \partial_{x_2} v_0^{(2,1)}(x_1, -(l_1 + l_2), t) + h_{2,2} \partial_{x_2} v_0^{(2,2)}(x_1, -(l_1 + l_2), t)}, \quad (4.16)$$

$$\eta_{2,2}(x_1, t) = \frac{h_{2,3} \partial_{x_2} v_0^{(2,3)}(x_1, -(l_1 + l_2), t)}{h_{2,3} \partial_{x_2} v_0^{(2,3)}(x_1, -(l_1 + l_2), t) + h_{2,4} \partial_{x_2} v_0^{(2,4)}(x_1, -(l_1 + l_2), t)}. \quad (4.17)$$

Relations (3.3), (3.11)-(3.14), (4.3), (4.4), (4.9), (4.10), (4.12)-(4.15) form *homogenized problem* for problem (2.4).

5 Operator formulation of the homogenized problem

To give appropriately the following definition of a weak solution of the homogenized problem, let us first introduce an anisotropic Sobolev space \mathbf{H} of multi-sheeted functions. A multi-sheeted function

$$\varphi := \left(\varphi^+, \varphi^{(0)}, \{ \varphi^{(1,m)} \}_{m=1}^2, \{ \varphi^{(2,m)} \}_{m=1}^4 \right) = \begin{cases} \varphi^+(x), & x \in \Omega_0, \\ \varphi^{(0)}(x), & x \in D_0, \\ \varphi^{(1,m)}(x), & x \in D_1, \quad m = 1, 2, \\ \varphi^{(2,m)}(x), & x \in D_2, \quad m = 1, 2, 3, 4, \end{cases}$$

belongs to \mathbf{H} if $\varphi^+ \in H^1(\Omega_0)$, $\{\varphi^{(i,m)}\}_{m=1}^{2i} \subset L^2(D_i)$, there exist weak derivatives $\{\partial_{x_2}\varphi^{(i,m)}\}_{m=1}^{2i} \subset L^2(D_i)$, $i = 0, 1, 2$, and

$$\begin{aligned}\varphi^+|_{I_0} &= \varphi^{(0)}|_{I_0}, & \varphi^{(0)}|_{I_1} &= \varphi^{(1,1)}|_{I_1} = \varphi^{(1,2)}|_{I_1}, \\ \varphi^{(1,1)}|_{I_2} &= \varphi^{(2,1)}|_{I_2} = \varphi^{(2,2)}|_{I_2}, & \varphi^{(1,2)}|_{I_2} &= \varphi^{(2,3)}|_{I_2} = \varphi^{(2,4)}|_{I_2}.\end{aligned}$$

Obviously, the space \mathbf{H} is continuously and densely embedded in the Hilbert space \mathbf{V} of multi-sheeted functions whose components belong to the corresponding L^2 -spaces, i.e., $\varphi \in \mathbf{V}$ if $\varphi^+ \in L^2(\Omega_0)$, $\{\varphi^{(i,m)}\}_{m=1}^{2i} \subset L^2(D_i)$, $i = 0, 1, 2$. The scalar products in \mathbf{V} and \mathbf{H} are defined as follows:

$$\begin{aligned}(\varphi, \psi)_{\mathbf{V}} &:= (\varphi^+, \psi^+)_{L^2(\Omega_0)} + \sum_{i=0}^2 \sum_{m=1}^{2i} (\varphi^{(i,m)}, \psi^{(i,m)})_{L^2(D_i)}, \\ (\varphi, \psi)_{\mathbf{H}} &:= (\varphi, \psi)_{\mathbf{V}} + (\nabla \varphi^+, \nabla \psi^+)_{L^2(\Omega_0)} + \sum_{i=0}^2 \sum_{m=1}^{2i} (\partial_{x_2} \varphi^{(i,m)}, \partial_{x_2} \psi^{(i,m)})_{L^2(D_i)}\end{aligned}$$

Recall that $\varphi^{(0,m)} = \varphi^{(0)}$ (see Remark 2.1). Since \mathbf{H} is continuously and densely embedded in \mathbf{V} , we can construct the Gelfand triple $\mathbf{H} \subset \mathbf{V} \subset \mathbf{H}^*$.

For almost every $t \in [0, T]$ we introduce an operator $\mathcal{A}(t) : \mathbf{H} \mapsto \mathbf{H}^*$ by the formula

$$\begin{aligned}\langle \mathcal{A}(t)\varphi, \psi \rangle &:= \int_{\Omega_0} \left(\nabla \varphi^+ \cdot \nabla \psi^+ + k(\varphi^+) \psi^+ \right) dx \\ &+ \sum_{i=0}^2 \sum_{m=1}^{2i} \int_{D_i} \left(h_{i,m} \partial_{x_2} \varphi^{(i,m)} \partial_{x_2} \psi^{(i,m)} + h_{i,m} k_i(\varphi^{(i,m)}) \psi^{(i,m)} + 2\delta_{\alpha_i,1} \kappa_i(\varphi^{(i,m)}) \psi^{(i,m)} \right) dx\end{aligned}$$

for all $\varphi, \psi \in L^2(0, T; \mathbf{H})$, and a linear functional $\mathbf{F}(t) \in \mathbf{H}^*$

$$\langle \mathbf{F}(t), \psi \rangle := \int_{\Omega_0} f_0 \psi^+ dx + 2 \sum_{i=0}^2 \delta_{\beta_i,1} \sum_{m=1}^{2i} \int_{D_i} g_0^{(i)} \psi^{(i,m)} dx.$$

Here $\langle \cdot, \cdot \rangle$ is the pairing of \mathbf{H}^* and \mathbf{H} , $\psi = \left(\psi^+, \psi^{(0)}, \{\psi^{(1,m)}\}_{m=1}^2, \{\psi^{(2,m)}\}_{m=1}^4 \right) \in L^2(0, T; \mathbf{H})$.

Now we can write down the homogenized problem in the form of the abstract Cauchy problem

$$\mathbf{v}' + \mathcal{A}(\mathbf{v}) = \mathbf{F} \quad \text{in } L^2(0, T; \mathbf{H}^*), \quad \mathbf{v}(0) = 0, \quad (5.1)$$

where $\mathbf{v} = \left(v^+, v^{(0)}, \{v^{(1,m)}\}_{m=1}^2, \{v^{(2,m)}\}_{m=1}^4 \right) \in L^2(0, T; \mathbf{H})$.

Definition 5.1. We say a multi-sheeted function

$$\mathbf{v} \in L^2(0, T; \mathbf{H}), \quad \text{with } \mathbf{v}' \in L^2(0, T; \mathbf{H}^*),$$

is a weak solution to the homogenized problem provided

$$\langle \mathbf{v}'(t), \mathbf{u} \rangle + \langle \mathcal{A}(t)\mathbf{v}, \mathbf{u} \rangle = \langle \mathbf{F}(t), \mathbf{u} \rangle \quad \forall \mathbf{u} \in \mathbf{H} \quad \text{and for a.e. } t \in (0, T),$$

and $\mathbf{v}|_{t=0} = 0$.

Remark 5.1. In view of the well-known properties of spaces $L^p(0, T; X)$ (see for instance [23]), the weak solution $\mathbf{v} \in C([0, T]; \mathbf{V})$, and thus the last equality in Definition 5.1 makes sense.

Theorem 5.1. There exists a unique weak multi-sheeted solution to the homogenized problem.

Proof. Let us show that for a.e. $t \in (0, T)$ the operator \mathcal{A} is bounded, strictly monotone, and hemicontinuous.

(1) Using (2.9), (6.17) and the definition of \mathcal{A} , we can prove the following inequality

$$|\langle \mathcal{A}(t)\varphi, \psi \rangle| \leq C_1(1 + \|\varphi\|_{\mathbf{H}})\|\psi\|_{\mathbf{H}} \quad \forall \varphi, \psi \in \mathbf{H},$$

from where it follow that \mathcal{A} is bounded.

(2) Operator \mathcal{A} is strongly monotone. Really, with the help of (2.8) we get

$$\begin{aligned} \langle \mathcal{A}\varphi - \mathcal{A}\psi, \varphi - \psi \rangle &\geq \int_{\Omega_0} \left(|\nabla(\varphi^+ - \psi^+)|^2 + c_1|\varphi^+ - \psi^+|^2 \right) dx \\ &+ \sum_{i=0}^2 \sum_{m=1}^{2i} \int_{D_i} \left(h_{i,m} |\partial_{x_2} \varphi^{(i,m)} - \partial_{x_2} \psi^{(i,m)}|^2 + (h_{i,m} + 2\delta_{\alpha_i,1}) c_1 |\varphi^{(i,m)} - \psi^{(i,m)}|^2 \right) dx \\ &\geq C_2 \|\varphi - \psi\|_{\mathbf{H}}^2 \quad \forall \varphi, \psi \in \mathbf{H}. \end{aligned}$$

(3) Operator \mathcal{A} is hemicontinuous. Indeed, the real valued function

$$[0, 1] \ni \tau \longmapsto \langle \mathcal{A}(\varphi + \tau \mathbf{v}), \psi \rangle$$

is continuous on $[0, 1]$ for all fixed $\varphi, \psi, \mathbf{v} \in \mathbf{H}$ due to the continuity of the functions $\{k, k_i, \kappa_i\}$, the right inequality in (2.10), and Lebesgue's dominated convergence theorem.

Thus, the realization $\mathcal{A} : L^2(0, T; \mathbf{H}) \mapsto L^2(0, T; \mathbf{H}^*)$ (we denote it by the same symbol) is bounded, monotone, and hemicontinuous, i.e., \mathcal{A} is type of M (see Lemma 2.1 [23]).

(4) Operator \mathcal{A} is coercive. Using (2.9), (2.10), and the Cauchy's inequality with δ ($ab \leq \delta a^2 + \frac{b^2}{4\delta}$, $a, b, \delta > 0$), we find

$$\begin{aligned} \int_0^T \langle \mathcal{A}(t)\varphi, \varphi \rangle dt &\geq C_3 \int_0^T \|\varphi\|_{\mathbf{H}}^2 dt - |k(0)| \int_0^T \int_{\Omega_0} |\varphi^+| dx dt \\ &- \sum_{i=0}^2 \sum_{m=1}^{2i} (h_{i,m} |k_i(0)| + 2\delta_{\alpha_i,1} |\kappa_i(0)|) \int_0^T \int_{D_i} |\varphi^{(i,m)}| dx dt \\ &\geq C_3 \int_0^T \|\varphi\|_{\mathbf{H}}^2 dt - \delta \int_0^T \|\varphi\|_{\mathbf{V}}^2 dt - C_4(\delta) \end{aligned}$$

for each $\varphi \in L^2(0, T; \mathbf{H})$. By selecting appropriate δ , we obtain the desired inequality for the coerciveness.

By Corollary 4.1 [23], problem (5.1) has a unique solution. \square

6 Asymptotic approximation

Let $\mathbf{v} = \left(v^+, v^{(0)}, \{v^{(1,m)}\}_{m=1}^2, \{v^{(2,m)}\}_{m=1}^4 \right) \in L^2(0, T; \mathbf{H})$ be a unique weak solution to the homogenized problem (5.1). With the help of \mathbf{v} , the junction-layer solutions $Z_1^{(0)}$ and $Z_2^{(0)}$ (see Proposition 3.1), the branch-layer solutions $\{Z_1^{(1)}, \Xi_1^{(1)}, \Xi_2^{(1)}\}$ (see Propositions 3.2, 3.3) in a neighborhood of the first branching zone I_1 , and the branch-layer solutions $\{Z_1^{(2,1)}, Z_1^{(2,2)}, \Xi_1^{(2,1)}, \Xi_2^{(2,1)}, \Xi_1^{(2,2)}, \Xi_2^{(2,2)}\}$ in a neighborhood of the second branching zone I_2 (see § 3.2.3), we define the leading terms in the asymptotic expansions (3.1), (3.2), (3.16), (3.20), (3.26), and (3.27).

An approximating function R_ε is constructed as the sum of the leading terms of the outer expansions (3.1), (3.2) and the inner expansion (3.16), (3.20), (3.26), (3.27) in neighborhoods of the joint zone I_0 and branching zones I_1, I_2 respectively, with the subtraction of the identical terms of their asymptotics because they are summed twice. As a result, we obtain

$$R_\varepsilon(x, t) = v^+(x, t) + \varepsilon \chi_0(x_2) \mathcal{N}_+^{(0)}\left(\frac{x}{\varepsilon}, x_1, t\right), \quad (x, t) \in \Omega_0 \times (0, T); \quad (6.1)$$

$$R_\varepsilon = v^{(0)}(x, t) + \varepsilon \left(Y_0\left(\frac{x_1}{\varepsilon}\right) \partial_{x_1} v^{(0)}(x, t) + \chi_0(x_2) \mathcal{N}_-^{(0)}\left(\frac{x}{\varepsilon}, x_1, t\right) + \chi_1(x_2) \mathcal{N}^{(1)}\left(\frac{x_1}{\varepsilon}, \frac{x_2+l_1}{\varepsilon}, x_1, t\right) \right), \quad (x, t) \in G_\varepsilon^{(0)} \times (0, T); \quad (6.2)$$

$$R_\varepsilon = v^{(1,m)}(x, t) + \varepsilon \left(Y_{1,m}\left(\frac{x_1}{\varepsilon}\right) \partial_{x_1} v^{(1,m)}(x, t) + \chi_1(x_2) \mathcal{N}_{1,m}^{(1)}\left(\frac{x_1}{\varepsilon}, \frac{x_2+l_1}{\varepsilon}, x_1, t\right) + \chi_2(x_2) \mathcal{N}_m^{(2)}\left(\frac{x_1}{\varepsilon}, \frac{x_2+l_1+l_2}{\varepsilon}, x_1, t\right) \right), \quad (x, t) \in G_\varepsilon^{(1,m)} \times (0, T), \quad m = 1, 2; \quad (6.3)$$

$$R_\varepsilon = v^{(2,m)}(x, t) + \varepsilon \left(Y_{2,m}\left(\frac{x_1}{\varepsilon}\right) \partial_{x_1} v^{(2,m)}(x, t) + \chi_2(x_2) \mathcal{N}_{2,m}^{(2)}\left(\frac{x_1}{\varepsilon}, \frac{x_2+l_1+l_2}{\varepsilon}, x_1, t\right) \right), \quad (x, t) \in G_\varepsilon^{(2,m)} \times (0, T), \quad m = 1, 2, 3, 4. \quad (6.4)$$

Here the function χ_0 is a smooth cutoff function such that $\chi_0(x_2) = 1$ for $|x_2| \leq \tau_0/2$, and $\chi_0(x_2) = 0$ for $|x_2| \geq \tau_0$, where τ_0 is sufficiently small number; $\chi_1(x_2) = \chi_0(x_2 + l_1)$, $\chi_2(x_2) = \chi_0(x_2 + l_1 + l_2)$, $x_2 \in \mathbb{R}$;

in (6.1)

$$\mathcal{N}_+^{(0)}(\xi, x_1, t) = \sum_{i=1}^2 (Z_i^{(0)}(\xi) - \delta_{i,2} \xi_2) \partial_{x_i} v^+(x_1, 0, t), \quad \xi = \frac{x}{\varepsilon},$$

where $\delta_{i,2}$ is the Kronecker delta;

in (6.2) $Y_0(\xi_1) = -\xi_1 + \frac{1}{2} + [\xi_1]$, where $[\xi_1]$ is the entire part of ξ_1 , and

$$\mathcal{N}_-^{(0)} = \left(Z_1^{(0)}(\xi) - Y_0(\xi_1) \right) \partial_{x_1} v^+(x_1, 0, t) + \left(Z_2^{(0)}(\xi) - \frac{\xi_2}{h_0} \right) \partial_{x_2} v^+(x_1, 0, t), \quad \xi = \frac{x}{\varepsilon},$$

$$\begin{aligned} \mathcal{N}^{(1)} &= \left(Z_1^{(1)}(\xi) - Y_0(\xi_1) \right) \partial_{x_1} v^{(0)}(x_1, -l_1, t) \\ &\quad + \left(\eta_1(x_1, t) \Xi_1^{(1)}(\xi) + (1 - \eta_1(x_1, t)) \Xi_2^{(1)}(\xi) - \xi_2 \right) \partial_{x_2} v^{(0)}(x_1, -l_1, t), \end{aligned}$$

$$\xi_1 = \frac{x_1}{\varepsilon}, \quad \xi_2 = \frac{x_2+l_1}{\varepsilon};$$

in (6.3) $Y_{1,m}(\xi_1) = -\xi_1 + b_{1,m} + [\xi_1]$, $m = 1, 2$, and

$$\begin{aligned} \mathcal{N}_{1,m}^{(1)}(\xi, x_1, t) &= \left(Z_1^{(1)}(\xi) - Y_{1,m}(\xi_1) \right) \partial_{x_1} v^{(0)}(x_1, -l_1, t) \\ &+ \left(\eta_1(x_1, t) (\Xi_1^{(1)}(\xi) - \delta_{1,m} \frac{h_0}{h_{1,1}} \xi_2) + (1 - \eta_1(x_1, t)) (\Xi_2^{(1)}(\xi) - \delta_{2,m} \frac{h_0}{h_{1,2}} \xi_2) \right) \partial_{x_2} v^{(0)}(x_1, -l_1, t), \\ \xi_1 &= \frac{x_1}{\varepsilon}, \quad \xi_2 = \frac{x_2 + l_1}{\varepsilon}, \end{aligned}$$

$$\begin{aligned} \mathcal{N}_m^{(2)}(\xi, x_1, t) &= \left(Z_1^{(2,m)}(\xi) - Y_{1,m}(\xi_1) \right) \partial_{x_1} v^{(1,m)}(x_1, -l_1 - l_2, t) \\ &+ \left(\eta_{2,m}(x_1, t) \Xi_1^{(2,m)}(\xi) + (1 - \eta_{2,m}(x_1, t)) \Xi_2^{(2,m)}(\xi) - \xi_2 \right) \partial_{x_2} v^{(1,m)}(x_1, -l_1 - l_2, t), \\ \xi_1 &= \frac{x_1}{\varepsilon}, \quad \xi_2 = \frac{x_2 + l_1 + l_2}{\varepsilon}, \quad m = 1, 2; \end{aligned}$$

in (6.4) $Y_{2,m}(\xi_1) = -\xi_1 + b_{2,m} + [\xi_1]$, $m = 1, 2, 3, 4$, and

$$\begin{aligned} \mathcal{N}_{2,m}^{(2)}(\xi, x_1, t) &= \left(Z_1^{(2,1)}(\xi) - Y_{2,m}(\xi_1) \right) \partial_{x_1} v^{(1,1)}(x_1, -l_1 - l_2, t) \\ &+ \left(\eta_{2,1}(x_1, t) (\Xi_1^{(2,1)}(\xi) - \delta_{1,m} \frac{h_{1,1}}{h_{2,1}} \xi_2) + (1 - \eta_{2,1}(x_1, t)) (\Xi_2^{(2,1)}(\xi) - \delta_{2,m} \frac{h_{1,1}}{h_{2,2}} \xi_2) \right) \partial_{x_2} v^{(1,1)}(x_1, -l_1 - l_2, t), \\ \xi_1 &= \frac{x_1}{\varepsilon}, \quad \xi_2 = \frac{x_2 + l_1 + l_2}{\varepsilon}, \quad m = 1, 2, \end{aligned}$$

$$\begin{aligned} \mathcal{N}_{2,m}^{(2)}(\xi, x_1, t) &= \left(Z_1^{(2,2)}(\xi) - Y_{2,m}(\xi_1) \right) \partial_{x_1} v^{(1,2)}(x_1, -l_1 - l_2, t) \\ &+ \left(\eta_{2,2}(x_1, t) (\Xi_1^{(2,2)}(\xi) - \delta_{3,m} \frac{h_{1,2}}{h_{2,3}} \xi_2) + (1 - \eta_{2,2}(x_1, t)) (\Xi_2^{(2,2)}(\xi) - \delta_{4,m} \frac{h_{1,2}}{h_{2,4}} \xi_2) \right) \partial_{x_2} v^{(1,2)}(x_1, -l_1 - l_2, t), \\ \xi_1 &= \frac{x_1}{\varepsilon}, \quad \xi_2 = \frac{x_2 + l_1 + l_2}{\varepsilon}, \quad m = 3, 4. \end{aligned}$$

Due to (4.3), (4.4), (4.9) and (4.12), the jumps $[R_\varepsilon]|_{Q_\varepsilon^{(i)}} = 0$, $i = 0, 1, 2$. This means that the approximating function R_ε belongs to $L^2(0, T; H^1(\Omega_\varepsilon))$.

Theorem 6.1. *Suppose that in addition to the assumptions made in Section 2, the following conditions hold: the function $f_0 \in C^1(\overline{\Omega_0} \times [0, T])$ and if some parameter $\beta_i = 1$ ($i = 0, 1, 2$), then the function $g_0^{(i)} \in C^1(\overline{D_i} \times [0, T])$ and it and its derivative with respect to x_2 vanish at $x_2 = -\sum_{n=0}^i l_n$ and $x_2 = -\sum_{n=0}^{i+1} l_n$.*

Then for any $\rho \in (0, 1)$ there exist positive constants C_0, ε_0 such that for all values $\varepsilon \in (0, \varepsilon_0)$ the difference between the solution v_ε to problem (2.4) and the approximating function R_ε defined by (6.1) – (6.4) satisfies the following estimate

$$\begin{aligned} &\max_{0 \leq t \leq T} \|R_\varepsilon(\cdot, t) - v_\varepsilon(\cdot, t)\|_{L^2(\Omega_\varepsilon)} + \|R_\varepsilon - v_\varepsilon\|_{L^2(0, T; H^1(\Omega_\varepsilon))} \\ &\leq C_0 \left(\varepsilon^{1-\rho} + \sum_{i=0}^2 \left(\varepsilon^{\alpha_i - 1 + \delta_{\alpha_i, 1}} + (1 - \delta_{\beta_i, 1}) \varepsilon^{\beta_i - 1} + \delta_{\beta_i, 1} \|g_\varepsilon^{(i)} - g_0^{(i)}\|_{L^2(G_\varepsilon^{(i)})} \right) \right). \quad (6.5) \end{aligned}$$

Proof. I. Residuals in the differential equations. Substituting R_ε in the differential equations of problem (2.4) instead of v_ε and calculating discrepancies with regard to problems (3.3), (3.11) – (3.13), we get

$$\begin{aligned} \partial_t R_\varepsilon - \Delta_x R_\varepsilon + k(R_\varepsilon) - f_0 &= k(R_\varepsilon) - k(v^+) + \varepsilon \chi_0(x_2) \partial_t \mathcal{N}_+^{(0)}(\xi, x_1, t) \Big|_{\xi=\frac{x}{\varepsilon}} \\ &\quad - \chi'_0(x_2) (\partial_{\xi_2} \mathcal{N}_+^{(0)}(\xi, x_1, t)) \Big|_{\xi=\frac{x}{\varepsilon}} - \varepsilon \partial_{x_2} (\chi'_0(x_2) \mathcal{N}_+^{(0)}(\frac{x}{\varepsilon}, x_1, t)) \\ &\quad - \chi_0(x_2) (\partial_{x_1 \xi_1}^2 \mathcal{N}_+^{(0)}(\xi, x_1, t)) \Big|_{\xi=\frac{x}{\varepsilon}} - \varepsilon \chi_0(x_2) \partial_{x_1} ((\partial_{x_1} \mathcal{N}_+^{(0)}(\xi, x_1, t)) \Big|_{\xi=\frac{x}{\varepsilon}}) \\ &\quad \text{in } \Omega_0 \times (0, T); \end{aligned} \quad (6.6)$$

$$\begin{aligned} \partial_t R_\varepsilon - \Delta_x R_\varepsilon + k_0(R_\varepsilon) &= k_0(R_\varepsilon) - k_0(v^{(0)}) - 2\delta_{\alpha_0,1} h_0^{-1} \kappa_0(v^{(0)}) + 2\delta_{\beta_0,1} h_0^{-1} g_0^{(0)} \\ &\quad - \varepsilon \partial_{x_1} \left(Y_0(\frac{x_1}{\varepsilon}) \partial_{x_1 x_1}^2 v^{(0)} \right) - \varepsilon \partial_{x_2} \left(Y_0(\frac{x_1}{\varepsilon}) \partial_{x_2 x_1}^2 v^{(0)} \right) \\ &\quad + \varepsilon \left(Y_0(\frac{x_1}{\varepsilon}) \partial_{x_1}^2 v^{(0)}(x, t) + \chi_0(x_2) \partial_t \mathcal{N}_-^{(0)}(\frac{x}{\varepsilon}, x_1, t) + \chi_1(x_2) \partial_t \mathcal{N}^{(1)}(\frac{x_1}{\varepsilon}, \frac{x_2+l_1}{\varepsilon}, x_1, t) \right) \\ &\quad - \chi'_0(x_2) (\partial_{\xi_2} \mathcal{N}_-^{(0)}(\xi, x_1, t)) \Big|_{\xi=\frac{x}{\varepsilon}} - \varepsilon \partial_{x_2} (\chi'_0(x_2) \mathcal{N}_-^{(0)}(\frac{x}{\varepsilon}, x_1, t)) \\ &\quad - \chi_0(x_2) (\partial_{x_1 \xi_1}^2 \mathcal{N}_-^{(0)}(\xi, x_1, t)) \Big|_{\xi=\frac{x}{\varepsilon}} - \varepsilon \chi_0(x_2) \partial_{x_1} ((\partial_{x_1} \mathcal{N}_-^{(0)}(\xi, x_1, t)) \Big|_{\xi=\frac{x}{\varepsilon}}) \\ &\quad - \chi'_1(x_2) (\partial_{\xi_2} \mathcal{N}^{(1)}(\xi, x_1, t)) \Big|_{\xi_1=\frac{x_1}{\varepsilon}, \xi_2=\frac{x_2+l_1}{\varepsilon}} - \varepsilon \partial_{x_2} (\chi'_1(x_2) \mathcal{N}^{(1)}(\frac{x_1}{\varepsilon}, \frac{x_2+l_1}{\varepsilon}, x_1, t)) \\ &\quad - \chi_1(x_2) (\partial_{x_1 \xi_1}^2 \mathcal{N}^{(1)}(\xi, x_1, t)) \Big|_{\xi_1=\frac{x_1}{\varepsilon}, \xi_2=\frac{x_2+l_1}{\varepsilon}} - \varepsilon \chi_1(x_2) \partial_{x_1} ((\partial_{x_1} \mathcal{N}^{(1)}(\xi, x_1, t)) \Big|_{\xi_1=\frac{x_1}{\varepsilon}, \xi_2=\frac{x_2+l_1}{\varepsilon}}) \\ &\quad \text{in } G_\varepsilon^{(0)} \times (0, T); \end{aligned} \quad (6.7)$$

and similar relations in $G_\varepsilon^{(1,m)} \times (0, T)$, ($m = 1, 2$) and $G_\varepsilon^{(2,m)} \times (0, T)$, ($m = 1, 2, 3, 4$) up to replacement of indices.

II. Residuals in the boundary and initial conditions. Obviously, $R_\varepsilon|_{t=0} = 0$. Also using (4.3), (4.4), (4.10), (4.11), (4.13), (4.15), (4.16) and (4.17), one can verify that

$$\begin{aligned} [\partial_{x_2} R_\varepsilon] \Big|_{Q_\varepsilon^{(0)}} &= -\varepsilon Y_0(\frac{x_1}{\varepsilon}) \partial_{x_1 x_2}^2 v^{(0)}(x_1, 0, t), \\ [\partial_{x_2} R_\varepsilon] \Big|_{Q_\varepsilon^{(1,m)}} &= \varepsilon \left(Y_0 \partial_{x_1 x_2}^2 v^{(0)}(x, t) - Y_{1,m} \partial_{x_1 x_2}^2 v^{(1,m)}(x, t) \right) \Big|_{x_2=-l_1}, \quad m = 1, 2, \\ [\partial_{x_2} R_\varepsilon] \Big|_{Q_\varepsilon^{(2,m)}} &= \varepsilon \left(Y_{1,1} \partial_{x_1 x_2}^2 v^{(1,1)}(x, t) - Y_{2,m} \partial_{x_1 x_2}^2 v^{(2,m)}(x, t) \right) \Big|_{x_2=-l_1-l_2}, \quad m = 1, 2, \\ [\partial_{x_2} R_\varepsilon] \Big|_{Q_\varepsilon^{(2,m)}} &= \varepsilon \left(Y_{1,2} \partial_{x_1 x_2}^2 v^{(1,1)}(x, t) - Y_{2,m} \partial_{x_1 x_2}^2 v^{(2,m)}(x, t) \right) \Big|_{x_2=-l_1-l_2}, \quad m = 3, 4, \end{aligned} \quad (6.8)$$

where $Q_\varepsilon^{(i,m)} = \partial G_\varepsilon^{(i,m)} \cap \{x_2 = -\sum_{n=1}^i l_n\}$.

Since $Z_1^{(0)}$ is odd in ξ_1 and $Z_2^{(0)}$ is even in ξ_1 (see Proposition 3.1), it is easy to check that $\partial_\nu R_\varepsilon = 0$ on $\partial\Omega_\varepsilon \cap \{x : x_2 \geq 0\}$. In additional, one can verify that

$$\begin{aligned} \partial_{x_2} R_\varepsilon \Big|_{\partial\Omega_\varepsilon \cap \{x_2=-l_1\}} &= \varepsilon Y_0(\frac{x_1}{\varepsilon}) \partial_{x_1 x_2}^2 v^{(0)}(x_1, -l_1, t), \\ \partial_{x_2} R_\varepsilon \Big|_{\partial\Omega_\varepsilon \cap \{x_2=-l_1-l_2\} \cap \partial G_\varepsilon^{(1,m)}} &= \varepsilon Y_{1,m}(\frac{x_1}{\varepsilon}) \partial_{x_1 x_2}^2 v^{(1,m)}(x_1, -l_1-l_2, t), \quad m = 1, 2, \\ \partial_{x_2} R_\varepsilon \Big|_{\partial\Omega_\varepsilon \cap \{x_2=-l_1-l_2-l_3\}} &= 0. \end{aligned} \quad (6.9)$$

Taking into account boundary conditions in problems (3.17), (3.21), (3.24), we find the values of $\partial_{x_1} R_\varepsilon$ on the vertical boundary of the branches:

$$\begin{aligned} \partial_{x_1} R_\varepsilon = \varepsilon \Big(& Y_0\left(\frac{x_1}{\varepsilon}\right) \partial_{x_1 x_1}^2 v^{(0)}(x, t) + \chi_0(x_2) (\partial_{x_1} \mathcal{N}_-^{(0)}(\xi, x_1, t)) \Big|_{\xi=\frac{x}{\varepsilon}} \\ & + \chi_1(x_2) (\partial_{x_1} \mathcal{N}^{(1)}(\xi, x_1, t)) \Big|_{\xi_1=\frac{x_1}{\varepsilon}, \xi_2=\frac{x_2+l_1}{\varepsilon}} \Big) \quad \text{on } \partial G_\varepsilon^{(0)} \cap \{x_2 \in (-l_1, 0)\}, \end{aligned} \quad (6.10)$$

$$\begin{aligned} \partial_{x_1} R_\varepsilon = \varepsilon \Big(& Y_{1,m}\left(\frac{x_1}{\varepsilon}\right) \partial_{x_1 x_1}^2 v^{(1,m)}(x, t) + \chi_1(x_2) (\partial_{x_1} \mathcal{N}_{1,m}^{(1)}(\xi, x_1, t)) \Big|_{\xi_1=\frac{x_1}{\varepsilon}, \xi_2=\frac{x_2+l_1}{\varepsilon}} \\ & + \chi_2(x_2) (\partial_{x_1} \mathcal{N}_m^{(2)}(\xi, x_1, t)) \Big|_{x_1=\frac{x_1}{\varepsilon}, x_2=\frac{x_2+l_1+l_2}{\varepsilon}} \Big) \\ & \text{on } \partial G_\varepsilon^{(1,m)} \cap \{x_2 \in (-l_1 - l_2, -l_1)\}, \quad m = 1, 2, \end{aligned} \quad (6.11)$$

$$\begin{aligned} \partial_{x_1} R_\varepsilon = \varepsilon \Big(& Y_{2,m}\left(\frac{x_1}{\varepsilon}\right) \partial_{x_1 x_1}^2 v^{(2,m)}(x, t) + \chi_2(x_2) (\partial_{x_1} \mathcal{N}_{2,m}^{(2)}(\xi, x_1, t)) \Big|_{x_1=\frac{x_1}{\varepsilon}, x_2=\frac{x_2+l_1+l_2}{\varepsilon}} \Big) \\ & \text{on } \partial G_\varepsilon^{(2,m)} \cap \{x_2 \in (-l_1 - l_2 - l_3, -l_1 - l_2)\}, \quad m = 1, 2, 3, 4. \end{aligned} \quad (6.12)$$

III. Residuals in the integral identity. Multiplying (6.6) and (6.7) for each indexes i and m with arbitrary function $\psi \in L^2(0, T; H^1(\Omega_\varepsilon))$, integrating by parts and taking (6.8)–(6.12) into account, we deduce

$$\int_{\Omega_\varepsilon} \partial_t R_\varepsilon \psi \, dx + \langle \mathcal{A}_\varepsilon(t) R_\varepsilon, \psi \rangle_\varepsilon = \int_{\Omega_0} f_0 \psi \, dx + \mathcal{F}_\varepsilon(\psi) \quad (6.13)$$

for a.e. $t \in (0, T]$. Subtracting the integral identity (2.11) from (6.13) and integrating over $t \in (0, \tau)$, where $\tau \in (0, T]$, we get

$$\begin{aligned} \int_0^\tau \Big(& \langle R'_\varepsilon - v'_\varepsilon, \psi \rangle_\varepsilon + \langle \mathcal{A}_\varepsilon(t) R_\varepsilon - \mathcal{A}_\varepsilon(t) v_\varepsilon, \psi \rangle_\varepsilon \Big) dt \\ & = \int_0^\tau \left(\mathcal{F}_\varepsilon(\psi) - \sum_{i=0}^2 \varepsilon^{\beta_i} \int_{\Upsilon_\varepsilon^{(i)}} g_\varepsilon^{(i)} \psi \, dx_2 \right) dt, \end{aligned} \quad (6.14)$$

where $\mathcal{F}_\varepsilon(\psi) = \sum_{j=1}^5 \mathcal{I}_j^\varepsilon(\psi)$ and (to short formulas we omit variables $\frac{x}{\varepsilon}, x, t$ in some places)

$$\mathcal{I}_1^\varepsilon(\psi) = \int_{\Omega_0} (k(R_\varepsilon) - k(v^+)) \psi \, dx + \sum_{i=0}^2 \sum_{m=1}^{2i} \int_{G_\varepsilon^{(i,m)}} (k_i(R_\varepsilon) - k_i(v^{(i,m)})) \psi \, dx,$$

$$\mathcal{I}_2^\varepsilon(\psi) = \sum_{i=0}^2 \sum_{m=1}^{2i} \left(\varepsilon^{\alpha_i} \int_{\Upsilon_\varepsilon^{(i,m)}} \kappa_i(R_\varepsilon) \psi \, dx_2 - 2\delta_{\alpha_i,1} h_{i,m}^{-1} \int_{G_\varepsilon^{(i,m)}} \kappa_i(v^{(i,m)}) \psi \, dx \right),$$

$$\mathcal{I}_3^\varepsilon(\psi) = 2 \sum_{i=0}^2 \sum_{m=1}^{2i} \delta_{\beta_i,1} h_{i,m}^{-1} \int_{G_\varepsilon^{(i,m)}} g_0^{(i)} \psi \, dx,$$

$$\begin{aligned}
\mathcal{I}_4^\varepsilon(\psi) = & \varepsilon \left(\int_{\Omega_0} \left(\chi_0(x_2) \partial_t \mathcal{N}_+^{(0)} \psi + \chi_0'(x_2) \mathcal{N}_+^{(0)} \partial_{x_2} \psi + \chi_0(x_2) (\partial_{x_1} \mathcal{N}_+^{(0)}(\xi, x_1, t)) \Big|_{\xi=\frac{x}{\varepsilon}} \partial_{x_1} \psi \right) dx \right. \\
& + \sum_{i=0}^2 \sum_{m=1}^{2i} \int_{G_\varepsilon^{(i,m)}} Y_{i,m}(\frac{x_1}{\varepsilon}) (\partial_{x_2 x_1}^2 v^{(i,m)} \partial_{x_2} \psi + \partial_{x_1 x_1}^2 v^{(i,m)} \partial_{x_1} \psi) dx \\
& + \int_{G_\varepsilon^{(0)}} \left(Y_0(\frac{x_1}{\varepsilon}) \partial_{tx_1}^2 v^{(0)} + \chi_0(x_2) \partial_t \mathcal{N}_-^{(0)}(\frac{x}{\varepsilon}, x_1, t) + \chi_1(x_2) \partial_t \mathcal{N}^{(1)}(\frac{x_1}{\varepsilon}, \frac{x_2+l_1}{\varepsilon}, x_1, t) \right) \psi dx \\
& + \int_{G_\varepsilon^{(0)}} \left(\chi_0(x_2) (\partial_{x_1} \mathcal{N}_-^{(0)}(\xi, x_1, t)) \Big|_{\xi=\frac{x}{\varepsilon}} + \chi_1(x_2) (\partial_{x_1} \mathcal{N}^{(1)}(\xi, x_1, t)) \Big|_{\xi_1=\frac{x_1}{\varepsilon}, \xi_2=\frac{x_2+l_1}{\varepsilon}} \right) \partial_{x_1} \psi dx \\
& + \int_{G_\varepsilon^{(0)}} \left(\chi_0'(x_2) \mathcal{N}_-^{(0)}(\frac{x}{\varepsilon}, x_1, t) + \chi_1'(x_2) \mathcal{N}^{(1)}(\frac{x_1}{\varepsilon}, \frac{x_2+l_1}{\varepsilon}, x_1, t) \right) \partial_{x_2} \psi dx \\
& + \sum_{m=1}^2 \int_{G_\varepsilon^{(1,m)}} \left(Y_{1,m}(\frac{x_1}{\varepsilon}) \partial_{tx_1}^2 v^{(1,m)} + \chi_1 \partial_t \mathcal{N}_{1,m}^{(1)}(\frac{x_1}{\varepsilon}, \frac{x_2+l_1}{\varepsilon}, x_1, t) + \chi_2 \partial_t \mathcal{N}_m^{(2)}(\frac{x_1}{\varepsilon}, \frac{x_2+l_1+l_2}{\varepsilon}, x_1, t) \right) \psi dx \\
& + \int_{G_\varepsilon^{(1,m)}} \left(\chi_1 (\partial_{x_1} \mathcal{N}_{1,m}^{(1)}(\xi, x_1, t)) \Big|_{\xi_1=\frac{x_1}{\varepsilon}, \xi_2=\frac{x_2+l_1}{\varepsilon}} + \chi_2 (\partial_{x_1} \mathcal{N}_m^{(2)}(\xi, x_1, t)) \Big|_{\xi_1=\frac{x_1}{\varepsilon}, \xi_2=\frac{x_2+l_1+l_2}{\varepsilon}} \right) \partial_{x_1} \psi dx \\
& + \int_{G_\varepsilon^{(1,m)}} \left(\chi_1'(x_2) \mathcal{N}_{1,m}^{(1)}(\frac{x_1}{\varepsilon}, \frac{x_2+l_1}{\varepsilon}, x_1, t) + \chi_2'(x_2) \mathcal{N}_m^{(2)}(\frac{x_1}{\varepsilon}, \frac{x_2+l_1+l_2}{\varepsilon}, x_1, t) \right) \partial_{x_2} \psi dx \\
& + \sum_{m=1}^4 \int_{G_\varepsilon^{(2,m)}} \left(Y_{2,m}(\frac{x_1}{\varepsilon}) \partial_{tx_1}^2 v^{(2,m)} + \chi_2(x_2) \partial_t \mathcal{N}_{2,m}^{(2)}(\frac{x_1}{\varepsilon}, \frac{x_2+l_1+l_2}{\varepsilon}, x_1, t) \right) \psi dx \\
& + \int_{G_\varepsilon^{(2,m)}} \chi_2(x_2) (\partial_{x_1} \mathcal{N}_{2,m}^{(2)}(\xi, x_1, t)) \Big|_{\xi_1=\frac{x_1}{\varepsilon}, \xi_2=\frac{x_2+l_1+l_2}{\varepsilon}} \partial_{x_1} \psi dx \\
& + \int_{G_\varepsilon^{(2,m)}} \chi_2'(x_2) \mathcal{N}_{2,m}^{(2)}(\frac{x_1}{\varepsilon}, \frac{x_2+l_1+l_2}{\varepsilon}, x_1, t) \partial_{x_2} \psi dx \Big), \\
\mathcal{I}_5^\varepsilon(\psi) = & - \int_{\Omega_0} \chi_0'(x_2) (\partial_{\xi_2} \mathcal{N}_+^{(0)}(\xi, x_1, t)) \Big|_{\xi=\frac{x}{\varepsilon}} \psi dx \\
& - \int_{G_\varepsilon^{(0)}} \left(\chi_0'(x_2) (\partial_{\xi_2} \mathcal{N}_-^{(0)}(\xi, x_1, t)) \Big|_{\xi=\frac{x}{\varepsilon}} + \chi_1'(x_2) (\partial_{\xi_2} \mathcal{N}^{(1)}(\xi, x_1, t)) \Big|_{\xi_1=\frac{x_1}{\varepsilon}, \xi_2=\frac{x_2+l_1}{\varepsilon}} \right) \psi dx \\
& - \sum_{m=1}^2 \int_{G_\varepsilon^{(1,m)}} \left(\chi_1' (\partial_{\xi_2} \mathcal{N}_{1,m}^{(1)}(\xi, x_1, t)) \Big|_{\xi_1=\frac{x_1}{\varepsilon}, \xi_2=\frac{x_2+l_1}{\varepsilon}} + \chi_2' (\partial_{\xi_2} \mathcal{N}_m^{(2)}(\xi, x_1, t)) \Big|_{\xi_1=\frac{x_1}{\varepsilon}, \xi_2=\frac{x_2+l_1+l_2}{\varepsilon}} \right) \psi dx
\end{aligned}$$

$$\begin{aligned}
& - \sum_{m=1}^4 \int_{G_\varepsilon^{(2,m)}} \chi_2'(x_2) (\partial_{\xi_2} \mathcal{N}_{2,m}^{(2)}(\xi, x_1, t)) \Big|_{\xi_1=\frac{x_1}{\varepsilon}, \xi_2=\frac{x_2+l_1+l_2}{\varepsilon}} \psi \, dx, \\
& \mathcal{I}_6^\varepsilon(\psi) = - \int_{\Omega_0} \chi_0(x_2) (\partial_{x_1 \xi_1}^2 \mathcal{N}_+^{(0)}(\xi, x_1, t)) \Big|_{\xi=\frac{x}{\varepsilon}} \psi \, dx \\
& - \int_{G_\varepsilon^{(0)}} \left(\chi_0(x_2) (\partial_{x_1 \xi_1}^2 \mathcal{N}_-^{(0)}(\xi, x_1, t)) \Big|_{\xi=\frac{x}{\varepsilon}} + \chi_1(x_2) (\partial_{x_1 \xi_1}^2 \mathcal{N}^{(1)}(\xi, x_1, t)) \Big|_{\xi_1=\frac{x_1}{\varepsilon}, \xi_2=\frac{x_2+l_1}{\varepsilon}} \right) \psi \, dx \\
& - \sum_{m=1}^2 \int_{G_\varepsilon^{(1,m)}} \left(\chi_1(\partial_{x_1 \xi_1}^2 \mathcal{N}_{1,m}^{(1)}(\xi, x_1, t)) \Big|_{\xi_1=\frac{x_1}{\varepsilon}, \xi_2=\frac{x_2+l_1}{\varepsilon}} + \chi_2(\partial_{x_1 \xi_1}^2 \mathcal{N}_m^{(2)}(\xi, x_1, t)) \Big|_{\xi_1=\frac{x_1}{\varepsilon}, \xi_2=\frac{x_2+l_1+l_2}{\varepsilon}} \right) \psi \, dx \\
& - \sum_{m=1}^4 \int_{G_\varepsilon^{(2,m)}} \chi_2(x_2) (\partial_{x_1 \xi_1}^2 \mathcal{N}_{2,m}^{(2)}(\xi, x_1, t)) \Big|_{\xi_1=\frac{x_1}{\varepsilon}, \xi_2=\frac{x_2+l_1+l_2}{\varepsilon}} \psi \, dx.
\end{aligned}$$

Let us estimate the right-hand side in (6.14). Due to the conditions (2.8) we have $|\mathcal{I}_1^\varepsilon(\psi)| \leq C_1 \varepsilon \|\psi\|_{L^2(\Omega_\varepsilon)}$. To estimate $|\mathcal{I}_2^\varepsilon(\psi)|$, we use special integral identities

$$\frac{\varepsilon h_{i,m}}{2} \int_{\Upsilon_\varepsilon^{(i,m)}} \phi \, dx_2 = \int_{G_\varepsilon^{(i,m)}} \phi \, dx - \varepsilon \int_{G_\varepsilon^{(i,m)}} Y_{i,m} \left(\frac{x_1}{\varepsilon} \right) \partial_{x_1} \phi \, dx \quad \forall \phi \in H^1(G_\varepsilon^{(i,m)}), \quad (6.15)$$

for $i \in \{0, 1, 2\}$, $m = \overline{1, 2i}$. To prove (6.15) it is enough to integrate by parts the last integral in (6.15). If $\alpha_i = 1$, then with the help of (6.15) we deduce

$$\begin{aligned}
& \left| \varepsilon^1 \int_{\Upsilon_\varepsilon^{(i,m)}} \kappa_i(R_\varepsilon) \psi \, dx_2 - 2h_{i,m}^{-1} \int_{G_\varepsilon^{(i,m)}} \kappa_i(v^{(i,m)}) \psi \, dx \right| \\
& \leq 2h_{i,m}^{-1} \int_{G_\varepsilon^{(i,m)}} \left| \kappa_i(R_\varepsilon) - \kappa_i(v^{(i,m)}) \right| |\psi| \, dx + \varepsilon \int_{G_\varepsilon^{(i,m)}} \left| Y_{i,m} \left(\frac{x_1}{\varepsilon} \right) \right| |\partial_{x_1} (\kappa_i(R_\varepsilon) \psi)| \, dx \\
& \leq C_2 \varepsilon \|\psi\|_{H^1(\Omega_\varepsilon)}. \quad (6.16)
\end{aligned}$$

In the last inequality we use (2.8), (6.17) and inequality $\max_{\mathbb{R}} |Y_{i,m}| \leq 1$. If $\alpha_i > 1$, then again with the help of (6.15) we get

$$\left| \varepsilon^{\alpha_i} \int_{\Upsilon_\varepsilon^{(i,m)}} \kappa_i(R_\varepsilon) \psi \, dx_2 \right| \leq C_3 \varepsilon^{\alpha_i-1} \|\psi\|_{H^1(\Omega_\varepsilon)}.$$

Therefore, $|\mathcal{I}_2^\varepsilon(\psi)| \leq C_4 \sum_{i=0}^2 \varepsilon^{\alpha_i-1+\delta_{\alpha_i,1}} \|\psi\|_{H^1(\Omega_\varepsilon)}$.

Similar, but now using (2.6) and (2.7), we can estimate

$$\begin{aligned}
& \left| \mathcal{I}_3^\varepsilon(\psi) - \sum_{i=0}^2 \varepsilon^{\beta_i} \int_{\Upsilon_\varepsilon^{(i)}} g_\varepsilon^{(i)} \psi \, dx_2 \right| \\
& \leq C_5 \|\psi\|_{H^1(\Omega_\varepsilon)} \sum_{i=0}^2 \left((1 - \delta_{\beta_i,1}) \varepsilon^{\beta_i-1} + \delta_{\beta_i,1} (\|g_\varepsilon^{(i)} - g_0^{(i)}\|_{L^2(G_\varepsilon^{(i,m)})} + \varepsilon) \right).
\end{aligned}$$

It is easy to see that $\mathcal{I}_4^\varepsilon(\psi)$ is of order $\mathcal{O}(\varepsilon)$. Thanks to the asymptotic estimates (3.18), (3.19), (3.22) – (3.25), all integrals in $\mathcal{I}_5^\varepsilon(\psi)$ are integrated over the support of the functions $\{\chi_i'\}_{i=0}^2$. Therefore, they are exponentially small.

Since the functions $\partial_{x_1\xi_1}^2 \mathcal{N}_+^{(0)}, \partial_{x_1\xi_1}^2 \mathcal{N}_-^{(0)}, \partial_{x_1\xi_1}^2 \mathcal{N}^{(1)}, \{\partial_{x_1\xi_1}^2 \mathcal{N}_{1,m}^{(1)}, \partial_{x_1\xi_1}^2 \mathcal{N}_m^{(2)}\}_{m=1}^2, \{\partial_{x_1\xi_1}^2 \mathcal{N}_{2,m}^{(2)}\}_{m=1}^4$ exponentially decrease as $|\xi_2| \rightarrow +\infty$ (see (3.18), (3.19), (3.22) – (3.25)), we deduce from Lemma 3.1 ([19]) that for any $\rho \in (0, 1)$ the integrals in $\mathcal{I}_6^\varepsilon(\psi)$ are of order $\mathcal{O}(\varepsilon^{1-\rho})$.

Regarding to the inequalities obtained above in this subsection, we conclude that for the right-hand side in (6.14) for every $\tau \in (0, T]$ the following inequality holds

$$\left| \int_0^\tau \left(\mathcal{F}_\varepsilon(\psi) - \sum_{i=0}^2 \varepsilon^{\beta_i} \int_{\Upsilon_\varepsilon^{(i)}} g_\varepsilon^{(i)} \psi dx_2 \right) dt \right| \leq C \|\psi\|_{L^2(0,T;H^1(\Omega_\varepsilon))} \times \left(\varepsilon^{1-\rho} + \sum_{i=0}^2 (\varepsilon^{\alpha_i-1+\delta_{\alpha_i,1}} + (1 - \delta_{\beta_i,1}) \varepsilon^{\beta_i-1} + \delta_{\beta_i,1} \|g_\varepsilon^{(i)} - g_0^{(i)}\|_{L^2(G_\varepsilon^{(i)})}) \right). \quad (6.17)$$

Putting $R_\varepsilon - v_\varepsilon$ instead ψ in (6.13) and taking into account that \mathcal{A}_ε is strictly monotone, we derive from (6.13) and (6.17) the estimate (6.5). \square

Remark 6.1. The constant C_0 in (6.5) depends on the following quantities:

$$\begin{aligned} & \sup_{(x_1,t) \in (0,a) \times (0,T)} |\partial_{tx_j}^2 v^+(x_1, 0, t)|, & \sup_{(x_1,t) \in (0,a) \times (0,T)} |\mathcal{D}^\alpha v^+(x_1, 0, t)|, \\ & \sup_{(x_1,t) \in (0,a) \times (0,T)} |\partial_{tx_j}^2 v^{(0)}(x_1, -l_1, t)|, & \sup_{(x_1,t) \in (0,a) \times (0,T)} |\mathcal{D}^\alpha v^{(0)}(x_1, -l_1, t)|, \\ & \sup_{(x_1,t) \in (0,a) \times (0,T)} |\partial_{tx_j}^2 v^{(1,m)}(x_1, -l_1 - l_2, t)|, & \sup_{(x_1,t) \in (0,a) \times (0,T)} |\mathcal{D}^\alpha v^{(1,m)}(x_1, -l_1 - l_2, t)|, \end{aligned}$$

$m = 1, 2, j = 1, 2$, and $\|\partial_{tx_1}^2 v_0^{(i,m)}\|_{L^2(D_i \times (0,T))}$, where $i \in \{0, 1, 2\}$, $m = \overline{1, 2i}$, $|\alpha| = \alpha_1 + \alpha_2 \leq 2$. Due to the assumptions for the functions f_0 and $\{g_0^{(i)}\}_{i=0}^2$ and condition (2.8) it follows from classical results on the smoothness of solutions to semilinear parabolic problems (see for instance §6 and §7 from [12, Sec. V]) that these quantities are bounded.

From Theorem 6.1 it follows directly the Corollary 2.1.

References

- [1] J.M. Arrieta, A.N. Carvalho, M.C. Pereira, R.P. Silva: Semilinear parabolic problems in thin domains with a highly oscillatory boundary. *Nonlinear Analysis*, 74 (2011) 5111–5132
- [2] D. Blanchard, A. Gaudiello: Homogenization of highly oscillating boundaries and reduction of dimension for monotone problem. *ESAIM: COCV*. 9 (2003) 449–460.
- [3] D. Blanchard, A. Gaudiello, J. Mossino: Highly oscillating boundaries and reduction of dimension: the critical case. *Anal. Appl. (Singap.)* 5 (2007) 137–163.

- [4] D. Blanchard, A. Gaudiello, T.A. Mel'nyk: Boundary homogenization and reduction of dimension in a Kirchhoff-Love plate. *SIAM J. Math. Anal.* 39 (2008) 1764–1787.
- [5] G.A. Chechkin, T.A. Mel'nyk: Spatial-skin effect for eigenvibrations of a thick cascade junction with "heavy" concentrated masses. *Math. Meth. Appl. Sci.* 37 (2014) 56–74.
- [6] D. Cioranescu, J. Saint Jean Paulin: Homogenization in open sets with holes. *J. Math. Anal. Appl.* 71 (1979) 590–607.
- [7] C. Conca, J.I. Diaz, A. Linan, C. Timofte: Homogenization in chemical reactive flows. *Electron. J. Differential Equations* 2004(40) (2004), 1–22.
- [8] U. De Maio, T. Durante, T.A. Mel'nyk: Asymptotic approximation for the solution to the Robin problem in a thick multi-level junction. *Mathematical Models and Methods in Applied Sciences*, 15 (2005) 1897–1921.
- [9] T. Durante, T.A. Mel'nyk: Homogenization of quasilinear optimal control problems involving a thick multilevel junction of type 3 : 2 : 1. *ESAIM: Control, Optimisation and Calculus of Variations*, 18 (2012) 583–610.
- [10] A.M. Il'in: Matching of asymptotic expansions of solutions of boundary value problems. *Translations of Mathematical Monographs*, 102. American Mathematical Society, Providence, RI, 1992.
- [11] V.A. Kondrat'ev, O.A. Oleinik: Boundary-value problems for partial differential equations in non-smooth domains. *Russian Math. Surveys*, 38(2) (1983) 1–86.
- [12] O.A. Ladyzhenskaya, V.A. Solonnikov, N.N. Uraltseva: Linear and quasi-linear equations of parabolic type, American Mathematical Society, Providence, RI, 1968.
- [13] M. Lenczner: Multiscale model for atomic force microscope array mechanical behavior. *Applied Physics Letters*, 90 (2007) 091908.
- [14] S.E. Lyshevshi: *Mems and Nems: Systems, Devices, and Structures*, CRC Press, Boca Raton, FL, 2002.
- [15] T.A. Mel'nyk: Homogenization of the Poisson equation in a thick periodic junction. *Zeitschrift für Analysis und ihre Anwendungen*, 18 (1999) 953–975.
- [16] T. A. Mel'nyk: Homogenization of a boundary-value problem with a nonlinear boundary condition in a thick junction of type 3:2:1. *Mathematical Models and Methods in Applied Sciences*, 31 (2008) 1005–1027.
- [17] T.A. Mel'nyk, G.A. Chechkin: Homogenization of a boundary-value problem in a thick cascade junction, *Journal of Mathematical Sciences*, 154(1) (2008) 50–77.

- [18] T.A. Mel'nyk, D.Yu. Sadovyj: Homogenization of boundary value problems in two-level thick junctions consisting of thin disks with rounded or sharp edges. *Journal of Mathematical Sciences*, 191(2) (2013) 254–280.
- [19] T.A. Mel'nyk, S.A. Nazarov: Asymptotics of the Neumann spectral problem solution in a domain of "thick comb". *Journal of Mathematical Sciences*, 85(6) (1997) 2326–2346.
- [20] S.A. Nazarov, B.A. Plamenevskii: *Elliptic problems in domains with piecewise smooth boundaries*, Berlin, Walter de Gruyter, 1994.
- [21] C.V. Pao: *Nonlinear parabolic and elliptic equations*, Plenum Press, New York, 1992.
- [22] M. Prizzi, K.P. Rybakowski: The effect of domain squeezing upon the dynamics of reaction-diffusion equations. *Journal of Differential Equations*, 173 (2001) 271–320.
- [23] R. E. Showalter, *Monotone operators in Banach space and nonlinear partial differential equations*, *Mathematical Surveys and Monographs*, Vol.49, American Mathematical Society, 1997.